11. Fourier Analysis

11.1 Fourier series

A function \( f(x) \) is called periodic if it is defined for all real \( x \) and if there is some positive number \( p \) such that
\[
f(x+p) = f(x).
\]
The number \( p \) is called a period of \( f(x) \). One example is illustrated in the following figure.
Trigonometric series

We know that sine and cosine functions are all periodic functions with period $2\pi$. Sine and cosine functions can be reproduced to variant periodic functions; for example, $1$, $\cos x$, $\sin x$, $\cos 2x$, $\sin 2x$, $\ldots$, $\cos nx$, $\sin nx$, $\ldots$

These functions have the period $2\pi/n$, and thus have different frequencies.

The series that will arise here will be of the form

$$a_0 + a_1 \cos x + b_1 \sin x + a_2 \cos 2x + b_2 \sin 2x + \ldots,$$

where $a_0$, $a_1$, $a_2$, $\ldots$, $b_1$, $b_2$, $b_3$, $\ldots$ are real constants. Such a series is called a trigonometric series, and the $a_n$ and $b_n$ are called the coefficients of the series. Using the summation sign, we may write this series as

$$a_0 + \sum_{n=1}^{\infty} (a_n \cos nx + b_n \sin nx).$$
Fourier series arise from the practical task of representing a given periodic function \( f(x) \) in terms of cosine and sine functions.

Let \( f(x) \) be a periodic function of period \( 2\pi \). \( f(x) \) can be represented by a trigonometric series,

\[
f(x) = a_0 + \sum_{n=1}^{\infty} (a_n \cos nx + b_n \sin nx) \quad \text{................................. (1)}
\]

where

\[
a_0 = \frac{1}{2\pi} \int_{-\pi}^{\pi} f(x) \, dx,
\]

\[
a_n = \frac{1}{\pi} \int_{-\pi}^{\pi} f(x) \cos nx \, dx, \quad n = 1, 2, \ldots
\]

\[
b_n = \frac{1}{\pi} \int_{-\pi}^{\pi} f(x) \sin nx \, dx, \quad n = 1, 2, \ldots
\]

The \( a_n \) and \( b_n \) coefficients mean the amounts of the components \( \cos nx \) and \( \sin nx \) for the original function \( f(x) \).

How to drive the coefficients \( a_0, a_n, \) and \( b_n \)?

Since

\[
\sin nx \sin mx = \frac{1}{2} [-\cos(n+m)x + \cos(n-m)x],
\]

\[
\cos nx \cos mx = \frac{1}{2} [\cos(n+m)x + \cos(n-m)x],
\]

\[
\sin nx \cos mx = \frac{1}{2} [\sin(n+m)x + \sin(n-m)x],
\]

\[
\int \sin x \, dx = -\cos x + c, \quad \text{and}
\]

\[
\int \cos x \, dx = \sin x + c,
\]

we have

\[
\int_{-\pi}^{\pi} \sin nx \sin mx \, dx = -\frac{1}{\pi} \int_{-\pi}^{\pi} \cos(n+m)x \, dx + \frac{1}{\pi} \int_{-\pi}^{\pi} \cos(n-m)x \, dx.
\]

If \( n \neq m \), then

\[
\int_{-\pi}^{\pi} \sin nx \sin mx \, dx = -A \sin(n+m)x \bigg|_{-\pi}^{\pi} + B \sin(n-m)x \bigg|_{-\pi}^{\pi} = 0;
\]

If \( n = m \), then

\[
\int_{-\pi}^{\pi} \sin nx \sin nx \, dx = -\frac{1}{4n} \sin 2nx \bigg|_{-\pi}^{\pi} + \frac{1}{2} \bigg|_{-\pi}^{\pi} = 0 + \pi = \pi.
\]
By the manner, we have

$$\int_{-\pi}^{\pi} \cos nx \cos mx \, dx = \frac{1}{2} \int_{-\pi}^{\pi} \cos((n + m)x) \, dx + \frac{1}{2} \int_{-\pi}^{\pi} \cos((n - m)x) \, dx$$

If \( n \neq m \), then \( \int_{-\pi}^{\pi} \cos nx \cos mx \, dx = A \sin(n + m)x \bigg|_{-\pi}^{\pi} + B \sin(n - m)x \bigg|_{-\pi}^{\pi} = 0 \);

If \( n = m \), then \( \int_{-\pi}^{\pi} \cos nx \cos nx \, dx = \frac{1}{4n} \sin 2nx \bigg|_{-\pi}^{\pi} + \frac{1}{2} x \bigg|_{-\pi}^{\pi} = 0 + \pi = \pi \).

$$\int_{-\pi}^{\pi} \sin nx \cos mx \, dx = \frac{1}{2} \int_{-\pi}^{\pi} \sin((n + m)x) \, dx + \frac{1}{2} \int_{-\pi}^{\pi} \sin((n - m)x) \, dx$$

If \( n \neq m \), then \( \int_{-\pi}^{\pi} \sin nx \cos mx \, dx = -A \cos(n + m)x \bigg|_{-\pi}^{\pi} - B \cos(n - m)x \bigg|_{-\pi}^{\pi} = 0 \);

If \( n = m \), then \( \int_{-\pi}^{\pi} \sin nx \cos nx \, dx = -\frac{1}{4n} \cos 2nx \bigg|_{-\pi}^{\pi} + 0 = 0 + 0 = 0 \).

Orthogonality of the trigonometric system

From the above formulas, we know that the trigonometric system:

\( \sin 0, \cos 0, \sin x, \cos x, \sin 2x, \cos 2x, \ldots, \sin nx, \cos nx, \ldots \)

are orthogonal on the interval \(-\pi \leq x \leq \pi\). That is,

$$\int_{-\pi}^{\pi} \cos nx \cos mx \, dx = 0, \text{ if } n \neq m,$$

$$\int_{-\pi}^{\pi} \sin nx \sin mx \, dx = 0, \text{ if } n \neq m,$$

$$\int_{-\pi}^{\pi} \cos nx \sin mx \, dx = 0, \text{ for any integers } n \text{ and } m,$$

$$\int_{-\pi}^{\pi} \cos nx \cos nx \, dx = \pi,$$

and

$$\int_{-\pi}^{\pi} \sin nx \sin nx \, dx = \pi.$$
Integrating both sides of Eq.(1) from \(-\pi\) to \(\pi\), we get
\[
\int_{-\pi}^{\pi} f(x) \, dx = \int_{-\pi}^{\pi} a_0 \, dx + \sum_{n=1}^{\infty} \left( a_n \int_{-\pi}^{\pi} \cos nx \, dx + b_n \int_{-\pi}^{\pi} \sin nx \, dx \right)
= 2\pi a_0 + 0 + 0.
\]
Thus \(a_0 = \frac{1}{2\pi} \int_{-\pi}^{\pi} f(x) \, dx\).

Similarly, we multiply Eq.(1) by \(\cos mx\) and integrate from \(-\pi\) to \(\pi\):
\[
\int_{-\pi}^{\pi} f(x) \cos mx \, dx
= a_0 \int_{-\pi}^{\pi} \cos mx \, dx + \sum_{n=1}^{\infty} \left( a_n \int_{-\pi}^{\pi} \cos nx \cos mx \, dx + b_n \int_{-\pi}^{\pi} \sin nx \cos mx \, dx \right)
= 0 + a_m \pi + 0 = a_m \pi.
\]
Thus \(a_m = \frac{1}{\pi} \int_{-\pi}^{\pi} f(x) \cos mx \, dx, \ m = 1, 2, ...\)

We multiply Eq.(1) by \(\sin mx\) and integrate from \(-\pi\) to \(\pi\):
\[
\int_{-\pi}^{\pi} f(x) \sin mx \, dx
= a_0 \int_{-\pi}^{\pi} \sin mx \, dx + \sum_{n=1}^{\infty} \left( a_n \int_{-\pi}^{\pi} \cos nx \sin mx \, dx + b_n \int_{-\pi}^{\pi} \sin nx \sin mx \, dx \right)
= 0 + 0 + b_m \pi = b_m \pi.
\]
Thus \(b_m = \frac{1}{\pi} \int_{-\pi}^{\pi} f(x) \sin mx \, dx, \ m = 1, 2, ...\)
Ex. Square wave

\[ f(x) = \begin{cases} -k, & \text{if } -\pi < x < 0 \\ k, & \text{if } 0 < x < \pi \end{cases} \]

with \( f(x + 2\pi) = f(x) \).

Answer.

\[
a_0 = \frac{1}{2\pi} \int_{-\pi}^{\pi} f(x) \, dx = 0
\]

\[
a_n = \frac{1}{\pi} \int_{-\pi}^{\pi} f(x) \cos nx \, dx = \frac{1}{\pi} \left[ \int_{-\pi}^{0} (-k) \cos nx \, dx + \int_{0}^{\pi} k \cos nx \, dx \right]
\]

\[
= \frac{1}{\pi} \left[ -k \frac{\sin nx}{n} \bigg|_{-\pi}^{0} + k \frac{\sin nx}{n} \bigg|_{0}^{\pi} \right] = 0
\]

\[
b_n = \frac{1}{\pi} \int_{-\pi}^{\pi} f(x) \sin nx \, dx = \frac{1}{\pi} \left[ \int_{-\pi}^{0} (-k) \sin nx \, dx + \int_{0}^{\pi} k \sin nx \, dx \right]
\]

\[
= \frac{1}{\pi} \left[ k \frac{\cos nx}{n} \bigg|_{-\pi}^{0} - k \frac{\cos nx}{n} \bigg|_{0}^{\pi} \right] = \frac{k}{n\pi} [\cos 0 - \cos(-n\pi) - \cos n\pi + \cos 0]
\]

\[
= \frac{2k}{n\pi} (1 - \cos n\pi)
\]

Problems of Section 11.1.
11.2 Functions of arbitrary period

- Fourier series of function with period $2L$

Let $f(x)$ be a periodic function of period $2L$. The Fourier series of $f(x)$ is described by

$$f(x) = a_0 + \sum_{n=1}^{\infty} \left( a_n \cos \frac{n\pi x}{L} + b_n \sin \frac{n\pi x}{L} \right)$$

where

$$(a_0 = \frac{1}{2L} \int_{-L}^{L} f(x) \, dx)$$

and

$$(a_n = \frac{1}{L} \int_{-L}^{L} f(x) \cos \frac{n\pi x}{L} \, dx, \quad n = 1, 2, \ldots)$$

and

$$(b_n = \frac{1}{L} \int_{-L}^{L} f(x) \sin \frac{n\pi x}{L} \, dx, \quad n = 1, 2, \ldots)$$

The formula means to change period from $2\pi$ to $2L$.

For example, $L = \pi/2$, the used functions are $\cos 2nx$ and $\sin 2nx$.

Problems of Section 11.2.

11.9 Fourier transform

- Complex Fourier series

By the Euler formula

$$e^{inx} = \cos nx + i \sin nx$$

$$e^{-inx} = \cos nx - i \sin nx.$$  

We have

$$\cos nx = \frac{1}{2} (e^{inx} + e^{-inx})$$

$$\sin nx = \frac{1}{2i} (e^{inx} - e^{-inx}).$$

Substituting $\cos nx$ and $\sin nx$ into

$$f(x) = a_0 + \sum_{n=1}^{\infty} (a_n \cos nx + b_n \sin nx)$$

to get

$$f(x) = a_0 + \sum_{n=1}^{\infty} \left( \frac{1}{2} (a_n - i b_n) e^{inx} + \frac{1}{2} (a_n + i b_n) e^{-inx} \right).$$
Rewriting as
\[ f(x) = c_0 + \sum_{n=1}^{\infty} (c_n e^{inx} + k_n e^{-inx}), \]
where
\[ c_n = \frac{1}{2} (a_n - i b_n) = \frac{1}{2\pi} \int_{-\pi}^{\pi} f(x) (\cos nx - i \sin nx) dx = \frac{1}{2\pi} \int_{-\pi}^{\pi} f(x) e^{-inx} dx, \]
\[ k_n = \frac{1}{2} (a_n + i b_n) = \frac{1}{2\pi} \int_{-\pi}^{\pi} f(x) (\cos nx + i \sin nx) dx = \frac{1}{2\pi} \int_{-\pi}^{\pi} f(x) e^{inx} dx, \]
\[ k_n = c_{-n}. \]
Thus
\[ f(x) = \sum_{n=-\infty}^{\infty} c_n e^{inx}, \]
where
\[ c_n = \frac{1}{2\pi} \int_{-\pi}^{\pi} f(x) e^{-inx} dx, \ n = \pm 1, \pm 2, \ldots \]

- **Complex Fourier series of function with period 2L**
  \[ f(x) = \sum_{n=-\infty}^{\infty} c_n \frac{1}{2L} \int_{-L}^{L} f(x) e^{-in\pi x/L} dx, \]
  where
  \[ c_n = \frac{1}{2L} \int_{-L}^{L} f(x) e^{-in\pi x/L} dx, \ n = \pm 1, \pm 2, \ldots \]

- **From Fourier series to the Fourier integral**
  If we let \( L \to \infty \), we can derive the Fourier integral
  \[ f(x) = \int_{-\infty}^{\infty} F(u) e^{i2\pi u x} du, \]
  where
  \[ F(u) = \int_{-\infty}^{\infty} f(x) e^{-i2\pi u x} dx. \]
**Fourier series**

\[ f(x) = a_0 + \sum_{n=1}^{\infty} (a_n \cos nx + b_n \sin nx) \]

where

\[ a_0 = \frac{1}{\pi} \int_{-\pi}^{\pi} f(x) \, dx, \]

\[ a_n = \frac{1}{\pi} \int_{-\pi}^{\pi} f(x) \cos nx \, dx, \quad \text{and} \]

\[ b_n = \frac{1}{\pi} \int_{-\pi}^{\pi} f(x) \sin nx \, dx. \]

**Fourier transforms**

Let \( f(x) \) be a continuous function of a real variable \( x \). The *Fourier transform* of \( f(x) \), denoted \( F(u) \), is defined by the equation

\[ \mathcal{F}\{f(x)\} = F(u) = \int_{-\infty}^{\infty} f(x) e^{-i2\pi ux} \, dx \]

where \( i = \sqrt{-1} \).

Given \( F(u) \), \( f(x) \) can be obtained by using the *inverse Fourier transform*

\[ \mathcal{F}^{-1}\{F(u)\} = f(x) = \int_{-\infty}^{\infty} F(u) e^{i2\pi ux} \, du \]

The above two equations are called the Fourier transform pair.

\[ F(u) = R(u) + i l(u), \] where \( R(u) \) and \( l(u) \) are the real and imaginary components of \( F(u) \), respectively.

In exponential form, \( F(u) = |F(u)| e^{i \phi(u)} \),

where \( |F(u)| = \left[ R^2(u) + l^2(u) \right]^{1/2} \) and \( \phi(u) = \tan^{-1} \left[ \frac{l(u)}{R(u)} \right] \).
The magnitude function $|F(u)|$ is called the **Fourier spectrum** of $f(x)$ and $\phi(u)$ is its **phase angle**. The square of the spectrum,

$$P(u) = |F(u)|^2 = R^2(u) + I^2(u),$$

is commonly referred to as the **power spectrum** of $f(x)$. The variable $u$ is called the **frequency variable**.

**Example**

Fourier transform transforms spatial-domain data into frequency-domain data, and inverse Fourier transform transforms frequency-domain data into spatial-domain data.

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**Example**

(a) A simple function,

$$F(u) = \int_{-\infty}^{\infty} f(x) e^{-i2\pi ux} \, dx$$

$$= \int_0^{X} A e^{-i2\pi ux} \, dx$$

$$= -\frac{A}{i2\pi u} \left[ e^{-i2\pi ux} \right]_0^X = -\frac{A}{i2\pi u} \left[ e^{-i2\pi uX} - 1 \right]$$

$$= \frac{A}{i2\pi u} \left[ e^{i\pi uX} - e^{-i\pi uX} \right] e^{-i\pi uX}$$

$$= A \frac{\sin(\pi uX)}{\pi u} e^{-i\pi uX}.$$
Two-dimensional Fourier transforms

\[ \mathcal{F}\{f(x,y)\} = F(u,v) = \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} f(x,y) e^{-i2\pi(ux+vy)} \, dx \, dy \]

\[ \mathcal{F}^{-1}\{F(u,v)\} = f(x,y) = \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} F(u,v) e^{i2\pi(ux+vy)} \, du \, dv \]

\[ |F(u,v)| = \left[ R^2(u,v) + l^2(u,v) \right]^{\frac{1}{2}} \]

\[ \phi(u,v) = \tan^{-1} \left[ \frac{l(u,v)}{R(u,v)} \right] \]

\[ P(u,v) = |F(u,v)|^2 = R^2(u,v) + l^2(u,v). \]

All definitions are the same as those of one-dimensional transform.

Example

(a) A 2-D function, (b) its Fourier spectrum, and (c) the spectrum displayed as an intensity function.
\[ F(u, v) = \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} f(x, y) e^{-i2\pi(ux+vy)} \, dx \, dy \]

\[ = A \int_{0}^{\infty} e^{-i2\pi ux} \, dx \int_{0}^{\infty} e^{-i2\pi vy} \, dy \]

\[ = A \left[ \frac{e^{-i2\pi ux}}{-i2\pi u} \right]_{0}^{\infty} \left[ \frac{e^{-i2\pi vy}}{-i2\pi v} \right]_{0}^{\infty} \]

\[ = \frac{A}{-i2\pi u} \left[ e^{-i2\pi ux} - 1 \right] \frac{1}{-i2\pi v} \left[ e^{-i2\pi vy} - 1 \right] \]

\[ = AXY \left[ \frac{\sin(\pi uX)}{\pi uX} \right] \left[ \frac{\sin(\pi vY)}{\pi vY} \right]. \]

The spectrum is

\[ |F(u, v)| = AXY \left| \frac{\sin(\pi uX)}{\pi uX} \right| \left| \frac{\sin(\pi vY)}{\pi vY} \right|. \]

**Three 2-D functions and their Fourier spectra**

Three spatial functions \( f(x, y) \).

The corresponding Fourier spectrum \(|F(u, v)|\).
The discrete Fourier transform

Suppose that a continuous \( f(x) \) is discretized into a sequence
\[
\{ f(x_0), f(x_0 + \Delta x), f(x_0 + 2\Delta x), \ldots, f(x_0 + [N-1]\Delta x) \}
\]
Defining \( f(x) = f(x_0 + x\Delta x) \), where \( x = 0, 1, 2, \ldots, N-1 \).
In other words, \( \{ f(0), f(1), f(2), \ldots, f(N-1) \} \), as shown in the following figure

![Sampling a continuous function.](image)

The 1-D discrete Fourier transform pair
\[
F(u) = \frac{1}{N} \sum_{x=0}^{N-1} f(x) e^{-i2\pi ux/N}
\]
for \( u = 0, 1, 2, \ldots, N-1 \), and
\[
f(x) = \sum_{u=0}^{N-1} F(u) e^{i2\pi ux/N}
\]
for \( x = 0, 1, 2, \ldots, N-1 \).

Note, \( \Delta u = \frac{1}{N\Delta x} \).

正餘弦函數 \( \sin x, \cos x \) 的週長是 \( 2\pi \);
\[
\sin \frac{2\pi}{N} x, \cos \frac{2\pi}{N} x \quad \text{相當於將週長轉換成} \ N.
\[
\sin \frac{2\pi u}{N} x, \cos \frac{2\pi u}{N} x \quad \text{相當於把週長轉換成} \ N/u.
\]
週長與頻率互為倒數 (reciprocal)；所以分解出來的正/餘弦
函數頻率為
\[
0, \frac{1}{N\Delta x}, \frac{2}{N\Delta x}, \frac{3}{N\Delta x}, \ldots, \frac{N-1}{N\Delta x}
\]
The 2-D discrete Fourier transform pair

\[ F(u, v) = \frac{1}{MN} \sum_{x=0}^{M-1} \sum_{y=0}^{N-1} f(x, y) e^{-i2\pi(ux/M+vy/N)} \]

for \( u = 0, 1, 2, ..., M-1 \), \( v = 0, 1, 2, ..., N-1 \), and

\[ f(x, y) = \sum_{u=0}^{M-1} \sum_{v=0}^{N-1} F(u, v) e^{i2\pi(ux/M+vy/N)} \]

for \( x = 0, 1, 2, ..., M-1 \) and \( y = 0, 1, 2, ..., N-1 \).

where \( \Delta u = \frac{1}{M\Delta x} \) and \( \Delta v = \frac{1}{N\Delta y} \).

If \( M = N \),

\[ F(u, v) = \frac{1}{N} \sum_{x=0}^{N-1} \sum_{y=0}^{N-1} f(x, y) e^{-i2\pi(ux+vy)/N} \quad \text{for} \quad u, v = 0, 1, 2, ..., N-1, \]

and

\[ f(x, y) = \frac{1}{N} \sum_{u=0}^{N-1} \sum_{v=0}^{N-1} F(u, v) e^{i2\pi(ux+vy)/N} \quad \text{for} \quad x, y = 0, 1, 2, ..., N-1. \]

The existence of discrete Fourier transform by showing

\[ F(u) = \frac{1}{N} \sum_{x=0}^{N-1} \left[ \sum_{r=0}^{N-1} F(r) e^{i2\pi rx/N} \right] e^{-i2\pi ux/N} \]

\[ = \frac{1}{N} \sum_{r=0}^{N-1} F(r) \left[ \sum_{x=0}^{N-1} e^{i2\pi rx/N} e^{-i2\pi ux/N} \right] \]

\[ = F(u), \]

Since the orthogonality condition

\[ \sum_{x=0}^{N-1} e^{i2\pi rx/N} e^{-i2\pi ux/N} = \begin{cases} N & \text{if} \quad r = u \\ 0 & \text{otherwise.} \end{cases} \]

Problems of Section 11.9.
11.11 Applications of Fourier transforms

A. Image enhancement

- Method
  \[ G(u,v) = H(u,v) F(u,v) \]

- Filter transfer function

a. Lowpass filtering

- Ideal filter (ILPF)  
  \[
  H(u,v) = \begin{cases} 
  1 & \text{if } (u^2 + v^2) \leq D_0 \\
  0 & \text{if } (u^2 + v^2) > D_0 
  \end{cases}
  \]

Example

(a) Original image and (b) the superimposed circles enclose 90, 93, 95, 99, and 99.5 percent of Fourier spectrum power, respectively.

(a) (b) (c) (d) (e) (f)
Butterworth filter

\[
H(u,v) = \frac{1}{1 + \left[ \frac{u^2 + v^2}{2D_0} \right]^{2n}}
\]

Two practical applications of lowpass filtering for image smoothing

(a) false contour and
(c) pepper and salt noise.

b. Lowpass filtering

- High-frequency emphasis
  In order to preserve the low-frequency components by adding a constant to a highpass filter transform function.

(a) original image,
(b) highpass Butterworth filter,
(c) high-frequency emphasis,
(d) high-frequency emphasis and histogram equalization.
11.12 Discrete cosine transform (DCT)

**Purpose**

Spatial domain: \( f(0), f(1), \ldots, f(N-1) \) 
variable \( x \)  
\( \downarrow \text{DCT} \)

frequency domain: \( c(0), c(1), \ldots, c(N-1) \) 
variable \( u \)

1-D discrete cosine (Fourier) transform

\[
c(u) = \alpha(u) \sum_{x=0}^{N-1} f(x) \cos \left(\frac{(2x+1)u\pi}{2N}\right), \quad u = 0, 1, 2, \ldots, N - 1,
\]

where \( \alpha(u) = \begin{cases} 
\sqrt{\frac{1}{N}}, & \text{for } u = 0 \\
\sqrt{\frac{2}{N}}, & \text{for } u = 1, 2, \ldots, N - 1.
\end{cases} \)

\( c(u), u = 0, 1, \ldots, N-1, \) are called the DCT of \( f(x) \).

1-D inverse discrete cosine transform (discrete cosine (Fourier) series)

\[
f(x) = \sum_{u=0}^{N-1} \alpha(u) c(u) \cos \left(\frac{(2x+1)u\pi}{2N}\right) \quad \text{for } x = 0, 1, 2, \ldots, N-1.
\]

Two-dimensional discrete cosine transform

\[
c(u, v) = \alpha(u) \alpha(v) \sum_{x=0}^{N-1} \sum_{y=0}^{N-1} f(x, y) \left[ \cos \left(\frac{(2x+1)u\pi}{2N}\right) \right] \left[ \cos \left(\frac{(2y+1)v\pi}{2N}\right) \right]
\]

for \( u, v = 0, 1, 2, \ldots, N-1 \).

Inverse 2-D DCT

\[
f(x, y) = \sum_{u=0}^{N-1} \sum_{v=0}^{N-1} \alpha(u) \alpha(v) c(u, v) \left[ \cos \left(\frac{(2x+1)u\pi}{2N}\right) \right] \left[ \cos \left(\frac{(2y+1)v\pi}{2N}\right) \right]
\]

for \( x, y = 0, 1, 2, \ldots, N-1 \).
One example of DCT basis functions ($N = 4$)

\[
\begin{bmatrix}
\cos \frac{(2x + 1)\pi}{2N} & \cos \frac{(2y + 1)\pi}{2N}
\end{bmatrix}
\]

Applications - 影像壓縮

(a) Lena 原始影像. (b) $CR = 32, \text{PSNR} = 33.43$.

(c) $CR = 64, \text{PSNR} = 30.62$. (d) $CR = 128, \text{PSNR} = 27.54$. 