3. Higher Order Linear ODEs

3.1 Homogeneous Linear ODEs

Definition of high-order linear DE

\[ y^{(n)} + p_{n-1}(x) y^{(n-1)} + \ldots + p_1(x) y' + p_0(x) y = r(x) \]

is a non-homogeneous linear equation

\[ y^{(n)} + p_{n-1}(x) y^{(n-1)} + \ldots + p_1(x) y' + p_0(x) y = 0 \]

is the corresponding homogeneous linear equation.

Solutions of the homogeneous equation

(i) general solution \( y(x) = c_1y_1(x) + c_2y_2(x) + \ldots + c_ny_n(x) \)

where \( c_i \)'s are arbitrary constant, \( y_i \)'s is a basis of solutions.

(ii) particular solution, if \( c_i \)'s are specific.

Theorem 1 (Superposition principle or linearity principle)

Any linear combination of homogeneous solutions is again a homogeneous solution.
Warning: Theorem 1 does not hold for non-homogeneous equation or a nonlinear equation.

Linear independent

\[ y_1(x), y_2(x), \ldots, y_n(x) \text{ are linear independent} \]
\[ \iff c_1 y_1 + c_2 y_2 + \ldots + c_n y_n = 0 \implies c_1 = c_2 = c_3 = \ldots = c_n = 0. \]
Otherwise, \( y_i \)'s are linear dependent.

Initial value problem

\[
\begin{align*}
  y^{(n)} + p_{n-1}(x) y^{(n-1)} + \ldots + p_1(x) y' + p_0(x) y &= 0 \\
  y(x_0) &= k_0 \\
  y'(x_0) &= k_1 \\
  \vdots \\
  y^{(n-1)}(x_0) &= k_{n-1}
\end{align*}
\]

\( y(x) \) on the interval \( I \).

Theorem 2 (Existence and uniqueness theorem)

If \( p_0(x), \ldots, p_{n-1}(x) \) are continuous functions on some open interval \( I \) and \( x_0 \) in \( I \), then the initial value problem has a unique solution \( y(x) \) on the interval \( I \).

Linear independence of solutions

\[
W(y_1, y_2, \ldots, y_n) = \begin{vmatrix} y_1 & y_2 & \ldots & y_n \\ y_1' & y_2' & \ldots & y_n' \\ \vdots & \vdots & \ddots & \vdots \\ y_1^{(n-1)} & y_2^{(n-1)} & \ldots & y_n^{(n-1)} \end{vmatrix}
\]

Theorem 3

If \( p_0(x), \ldots, p_{n-1}(x) \) are continuous, then \( n \) solutions \( y_1(x), \ldots, y_n(x) \),

(a) \( y_i \)'s are linear dependent \( \iff W[y_i \text{'s}] = 0 \).

(b) if \( W[x_0] = 0 \), \( x_0 \) in \( I \) \( \implies W[x] \equiv 0 \) for all \( x \) in \( I \) if \( W[x_0] \neq 0 \), \( x_0 \) in \( I \) \( \implies W[x] \neq 0 \) for all \( x \) in \( I \).

Proof: The same as the proof of Theorem 2 of Section 2.6.
A general solution of homogeneous equation

Theorem 4 and 5 (Existence of a general solution)
If \( p_0(x), p_1(x), \ldots, p_{n-1}(x) \) are continuous on some open interval \( I \), then the homogeneous equation has a general solution of the form
\[
y(x) = c_1 y_1 + c_2 y_2 + \ldots + c_n y_n \quad \text{on } I_0.
\]

Problems of Section 3.1.

3.2 Homogeneous linear ODEs with constant coefficients

For a homogeneous equation
\[
y^{(n)} + a_{n-1}y^{(n-1)} + \ldots + a_1 y' + a_0y = 0 \quad \text{.......................... (1)}
\]
substituting \( y(x) = e^{\lambda x} \) into Eq.(1) to obtain the characteristic equation
\[
\lambda^n + a_{n-1}\lambda^{n-1} + \ldots + a_1 \lambda + a_0 = 0 \quad \text{.......................... (2)}
\]

Consider the roots of the characteristic equation

Case 1. Distinct real roots
If Eq.(2) has \( n \) real unequal roots \( \lambda_1, \lambda_2, \lambda_3, \ldots, \lambda_n \), then the \( n \) solutions \( y_1 = e^{\lambda_1 x}, y_2 = e^{\lambda_2 x}, \ldots, y_n = e^{\lambda_n x} \) constitute a basis for all \( x \), and the corresponding general solution of Eq.(1) is
\[
y = c_1 y_1 + c_2 y_2 + \ldots + c_n y_n.
\]
Theorem 1. If all $\lambda_i$'s are different, then $e^{\lambda_i x}$'s are linearly independent.

Proof.

$$\begin{vmatrix}
e^{\lambda_1 x} & e^{\lambda_2 x} & \cdots & e^{\lambda_n x} \\
\lambda_1 e^{\lambda_1 x} & \lambda_2 e^{\lambda_2 x} & \cdots & \lambda_n e^{\lambda_n x} \\
\vdots & \vdots & \ddots & \vdots \\
\lambda_1^{n-1} e^{\lambda_1 x} & \cdots & \cdots & \lambda_n^{n-1} e^{\lambda_n x}
\end{vmatrix}$$

$$W = e^{(\lambda_1+\lambda_2+\ldots+\lambda_n)x} \begin{vmatrix} 1 & 1 & \cdots & 1 \\
\lambda_1 & \lambda_2 & \cdots & \lambda_n \\
\vdots & \vdots & \ddots & \vdots \\
\lambda_1^{n-1} & \cdots & \cdots & \lambda_n^{n-1}
\end{vmatrix}$$

The determinant is called Vandermonde or Cauchy determinant

$$= e^{(\lambda_1+\ldots+\lambda_n)x} (-1)^{\frac{n(n-1)}{2}} (\lambda_1 - \lambda_2) (\lambda_1 - \lambda_3) \cdots (\lambda_1 - \lambda_n)$$

$$\frac{(\lambda_2 - \lambda_3) \cdots (\lambda_2 - \lambda_n)}{(\lambda_3 - \lambda_4) \cdots (\lambda_3 - \lambda_n)}$$

$$\vdots$$

$$= \frac{(\lambda_{n-1} - \lambda_n)}{}$$

$$\neq 0.$$

Case 2. Simple complex roots (有一對共軛複數根)

If complex roots occur, they must occur in conjugate pairs since the coefficients of Eq.(1) are real. If $\lambda = \nu + i \omega$ is a simple root of Eq.(2) so is the conjugate $\lambda = \nu - i \omega$, and two corresponding linearly independent solution are $y_1 = e^{\nu x} \cos \omega x$, $y_2 = e^{\nu x} \sin \omega x$.

Ex.2.

$$y''' - 2 y'' + 2 y' = 0.$$

$$\Rightarrow \lambda^3 - 2 \lambda^2 + 2 \lambda = 0.$$

$$\Rightarrow \lambda_1 = 0, \lambda_2 = 1 + i, \lambda_3 = 1 - i.$$

$$\Rightarrow y_1 = 1, y_2 = e^{x} \cos x, y_3 = e^{x} \sin x.$$
Case 3. Multiple real roots

- If a real double root occurs, say $\lambda_1 = \lambda_2$, then taking $y_1 = e^{\lambda_1 x}$ and $y_2 = x e^{\lambda_1 x}$.

- If a triple root occurs, say $\lambda_1 = \lambda_2 = \lambda_3$, then taking $y_1 = e^{\lambda_1 x}$, $y_2 = x e^{\lambda_1 x}$, $y_3 = x^2 e^{\lambda_1 x}$.

- If $\lambda_1$ is a root of order $m$, then $m$ corresponding linearly independent solutions are $e^{\lambda_1 x}$, $x e^{\lambda_1 x}$, $\ldots$, $x^{m-1} e^{\lambda_1 x}$.

How to obtain the solutions $e^{\lambda_1 x}$, $x e^{\lambda_1 x}$, $\ldots$, $x^{m-1} e^{\lambda_1 x}$?

Let $\lambda_1$ be an $m$th order root of the polynomial, and let $\lambda_{m+1}$, $\ldots$, $\lambda_n$ be other roots, all different to $\lambda_1$, when $m < n$.

$$L[e^{\lambda x}] = (\lambda^n + a_{n-1} \lambda^{n-1} + \ldots + a_0) e^{\lambda x}$$

Differentiating the equation w.r.t. $\lambda$

$$L[\frac{\partial}{\partial \lambda} e^{\lambda x}] = \frac{\partial}{\partial \lambda} [((\lambda - \lambda_1)^m h(\lambda) e^{\lambda x}]$$

$$L[\frac{\partial}{\partial \lambda} e^{\lambda x}] = m(\lambda - \lambda_1)^{m-1} h(\lambda) e^{\lambda x} + (\lambda - \lambda_1)^m \frac{\partial}{\partial \lambda} [h(\lambda) e^{\lambda x}]$$

$$L[xe^{\lambda x}] = m(\lambda - \lambda_1)^{m-1} h(\lambda) e^{\lambda x} + (\lambda - \lambda_1)^m \frac{\partial}{\partial \lambda} [h(\lambda) e^{\lambda x}]$$

$\Rightarrow$ $L[xe^{\lambda x}] = 0$ if $m \geq 2$

$\Rightarrow xe^{\lambda x}$ is a solution of Eq.(1).

We can repeat this step and produce $x^2 e^{\lambda_1 x}$, $\ldots$, $x^{m-1} e^{\lambda_1 x}$, by another $(m - 2)$ such differentiation with respect to $\lambda$. 
There is no longer produce solution, so we get precisely the solutions \(e^{\lambda_1 x}, x e^{\lambda_1 x}, \ldots, x^{m-1} e^{\lambda_1 x}\).

**Case 4. Multiple complex roots (多對相同共軛樹根)**

If \(\lambda = v + iw\) is a complex double root, so is the conjugate \(\bar{\lambda} = v - iw\). The corresponding linearly independent solutions are \(e^{vx}\cos wx, e^{vx}\sin wx, xe^{vx}\cos wx, xe^{vx}\sin wx\).

If \(\lambda\) is a complex triple root, the corresponding linearly independent solutions are

\(e^{vx}\cos wx, e^{vx}\sin wx, xe^{vx}\cos wx, xe^{vx}\sin wx, x^2 e^{vx}\cos wx, x^2 e^{vx}\sin wx\).

**Problems of Section 3.2.**

### 3.3 Nonhomogeneous linear ODEs

\[y^{(n)} + p_{n-1} y^{(n-1)} + \ldots + p_1(x) y' + p_0(x) y = r(x) \quad \ldots \ldots \ldots \ldots \quad (1)\]

The corresponding homogeneous equation

\[y^{(n)} + p_{n-1} y^{(n-1)} + \ldots + p_1(x) y' + p_0(x) y = 0 \quad \ldots \ldots \ldots \ldots \quad (2)\]

**Theorem 1.** Let \(y_1\) and \(y_2\) be arbitrary solution of Eq.(1), and \(y_h\) is an arbitrary solution of Eq.(2)

(a) \((y_1 - y_2)\) is a solution of Eq.(2)

(b) \((y_1 + y_h)\) is a solution of Eq.(1).

**A general solution of the non-homogeneous equation (1) on some open interval \(I\) is a solution of the form**

\[y(x) = y_h(x) + y_p(x) \quad \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \quad (3)\]

where \(y_h(x) = c_1 y_1(x) + \ldots + c_n y_n(x)\) is a general solution of the homogeneous equation (2).
A particular solution of Eq.(1) on \( I \) is a solution obtained from Eq.(3) by assigning specific values to the arbitrary constants \( c_1, c_2, \ldots, c_n \) in \( y_h(x) \).

**Theorem 2 (General solution)**

If all coefficients \( p_i(x) \)'s, \( 0 \leq i \leq n - 1 \), and \( r(x) \) are continuous on some open interval \( I \), then Eq.(1) has a general solution on \( I \).

**Initial value problem**

\[
\begin{align*}
y^{(n)} + p_{n-1}(x) y^{(n-1)} + \cdots + p_0 y &= r(x) \\
y(x_0) &= k_0 \\
y'(x_0) &= k_1 \\
&\quad \vdots \\
y^{(n-1)}(x_0) &= k_{n-1}.
\end{align*}
\]

**Theorem 3** If \( p_i(x) \)'s and \( r(x) \) are continuous on some open interval \( I \). Then Eq.(4) has a unique solution on \( I \).

**Method of undetermined coefficients**

\[
y^{(n)} + p_{n-1} y^{(n-1)} + \cdots + p_0(x) y = r(x)
\]

**General solution is** \( y = y_h(x) + y_p(x) \),

where \( y_h(x) \) is the general solution of the corresponding homogeneous equation. \( y_p(x) \) is the particular solution of the nonhomogeneous equation. Find \( y_p(x) \) by choosing a form similar to that of \( r(x) \) and determining the associated unknown coefficients.
How to choose the form of $y_p(x)$?
(a) Basis rule (as in Section 2.7)
(b) Modification rule
   if $cr(x)$ is a solution of the corresponding homogeneous equation (2), then multiply $y_p(x)$ by $x^k$, where $k$ is the smallest positive integer such that no term of $x^k y_p(x)$ is a solution of Eq.(2).
(c) Sum rule
   If $r(x)$ is linear combination of functions listed in the left column of listed Table 2.1 of Section 2.7, then $y_p(x) = $ linear combination of the corresponding functions in the right column of Table 2.1.

Solving initial value problem
step 1. solving $y_h(x)$
step 2. solving $y_p(x)$
step 3. taking $y_h(x) + y_p(x)$ to satisfy the initial conditions.

Method of variation of parameters
\[
y^{(n)} + p_{n-1}(x) y^{(n-1)} + \ldots + p_0(x) y = r(x) \quad \text{................................. (1)}
\]
\[
y^{(n)} + p_{n-1}(x) y^{(n-1)} + \ldots + p_0(x) y = 0 \quad \text{................................. (2)}
\]
\[
y_p(x) = y_1(x) \int \frac{W_1(x)}{W(x)} r(x) \, dx
+ y_2(x) \int \frac{W_2(x)}{W(x)} r(x) \, dx
+ \ldots
+ y_n(x) \int \frac{W_n(x)}{W(x)} r(x) \, dx
\]

Here, $y_1, y_2, \ldots, y_n$ is a basis of solution of the corresponding homogeneous equation on $I$, $W$ is the Wronskian of $y_i$'s, and $W_j$ ($j = 1, \ldots, n$) is obtained from $W$ by replacing the $j$th column of $W$ by the column $[0 \ldots 0 1]^T$. 
\[ W = \begin{vmatrix} y_1 & y_2 & \cdots & y_n \\ y_1' & \cdots & \cdots & y_n' \\ \vdots & \vdots & \ddots & \vdots \\ y_1^{(n-1)} & \cdots & \cdots & y_n^{(n-1)} \end{vmatrix} \Rightarrow W_j = \begin{vmatrix} y_1 & \cdots & y_{j-1} & 0 & y_{j+1} & \cdots & y_n \\ y_1' & \cdots & y_{j-1}' & 0 & y_{j+1}' & \cdots & y_n' \\ \vdots & \vdots & \ddots & \vdots & \vdots & \ddots & \vdots \\ y_1^{(n-1)} & \cdots & y_{j-1}^{(n-1)} & 1 & y_{j+1}^{(n-1)} & \cdots & y_n^{(n-1)} \end{vmatrix} \]

The proof is the same as the idea in Section 2.10.

\[ y^{(n)} + p_{n-1}(x) y^{(n-1)} + \ldots + p_1(x) y' + p_0(x) y = 0. \]

\[ \equiv L \left[ y \right] = 0, \quad y = c_1 y_1 + \ldots + c_n y_n. \]

Taking \( y_p = u_1(x) y_1 + \ldots + u_n(x) y_n \), then to solve \( u_i(x)'s \).

We need \( n \) conditions to solve \( u_i(x)'s \). Now we just have one condition

\[ y_p^{(n)} + p_{n-1} y_p^{(n-1)} + \ldots + p_0 y_p = r(x), \]

To derive other \( n - 1 \) conditions,

\[ y_p' = (u_1'y_1 + \ldots + u_n'y_n) + (u_1'y_1' + \ldots + u_n'y_n') \]

Take \( u_1'y_1 + \ldots + u_n'y_n = 0 \) as the 1st condition,

\[ y_p'' = (u_1'y_1' + \ldots + u_n'y_n') + (u_1'y_1'' + \ldots + u_n'y_n'') \]

Take \( u_1'y_1' + \ldots + u_n'y_n' = 0 \) as the 2nd condition.

\[ \ldots \]

and so on, until

\[ y_p^{(n-1)} = (u_1'y_1^{(n-2)} + \ldots + u_n'y_n^{(n-2)}) + (u_1'y_1^{(n-1)} + \ldots + u_n'y_n^{(n-1)}) \]

Take \( u_1'y_1^{(n-2)} + \ldots + u_n'y_n^{(n-2)} = 0 \) as the \( (n-1) \) th condition.

Differentiating \( y_p^{(n-1)}(x) \) to get

\[ y_p^{(n)} = (u_1'y_1^{(n)} + \ldots + u_n'y_n^{(n)}) + (u_1'y_1^{(n-1)} + \ldots + u_n'y_n^{(n-1)}) \]

Substituting all \( y_p^{(j)} \), \( 0 \leq j \leq n \) into Eq.(1) to get

\[ (u_1'y_1^{(n)} + \ldots + u_n'y_n^{(n)}) + (u_1'y_1^{(n-1)}) + \ldots + u_n'y_n^{(n-1)} + \]

\[ p_{n-1} (u_1'y_1^{(n-1)} + \ldots + u_n'y_n^{(n-1)}) + \]

\[ \vdots \]

\[ p_0 (u_1'y_1 + \ldots + u_n'y_n) = r(x) \]
\[ u_1 \left( y_1^{(n)} + p_{n-1} y_1^{(n-1)} + \ldots + p_0 y_1 \right) + \]
\[ u_2 \left( y_2^{(n)} + p_{n-1} y_2^{(n-1)} + \ldots + p_0 y_2 \right) + \]
\[ \vdots \]
\[ u_n \left( y_n^{(n)} + p_{n-1} y_n^{(n-1)} + \ldots + p_0 y_n \right) + \]
\[ u_1'y_1^{(n-1)} + \ldots + u_n'y_n^{(n-1)} = r(x) \]

\[ u_1'y_1^{(n-1)} + \ldots + u_n'y_n^{(n-1)} = r(x) \quad \text{(the last condition)} \]

\[
\begin{align*}
\begin{bmatrix}
u_1' \\
u_2' \\
\vdots \\
u_n'
\end{bmatrix} &=
\begin{bmatrix}
0 \\
0 \\
\vdots \\
r
\end{bmatrix} \\
\begin{bmatrix}
ta_{i1} & a_{i2} & \ldots & a_{in} \\
\vdots & \vdots & \ddots & \vdots \\
ta_{n1} & a_{n2} & \ldots & a_{nn}
\end{bmatrix} &= t
\begin{bmatrix}
a_{i1} & a_{i2} & \ldots & a_{in} \\
\vdots & \vdots & \ddots & \vdots \\
a_{n1} & a_{n2} & \ldots & a_{nn}
\end{bmatrix}
\end{align*}
\]

\[ u_j' = \frac{W_j r}{W} \quad \text{(Cramer's rule)} \]

\[ u_j = \int \frac{W_j}{W} r \, dx \]

\[ y_p(x) = \sum y_i(x) \int \frac{W_i}{W} r(x) \, dx. \]

Problems of Section 3.3.