5. Eigenvalues and Eigenvectors

5.1 Eigenvectors and eigenvalues

- Definition

  An eigenvector (固有向量) of an $n \times n$ matrix $\boldsymbol{A}$ is a nonzero vector $\boldsymbol{x}$ such that $\boldsymbol{Ax} = \lambda\boldsymbol{x}$ for some scalar λ. Scalar λ is called an eigenvalue (固有值) of $\boldsymbol{A}$ if there is a non-trivial solution $\boldsymbol{x}$ for $\boldsymbol{Ax} = \lambda\boldsymbol{x}$; such that $\boldsymbol{x}$ is called an eigenvector corresponding to λ .

Ex.2.

(*i*) Is $\begin{bmatrix} 6 \\ -5 \end{bmatrix}$ an eigenvector of $\begin{bmatrix} 1 & 6 \\ 5 & 2 \end{bmatrix}$?

(*ii*) Is $\begin{bmatrix} 3 \\ -2 \end{bmatrix}$ an eigenvector of $\begin{bmatrix} 1 & 6 \\ 5 & 2 \end{bmatrix}$?

Answer.

$$(i)\ \begin{bmatrix} 1 & 6 \\ 5 & 2 \end{bmatrix}\begin{bmatrix} 6 \\ -5 \end{bmatrix} = \begin{bmatrix} -24 \\ 20 \end{bmatrix} = -4\begin{bmatrix} 6 \\ -5 \end{bmatrix} \Rightarrow \begin{bmatrix} 6 \\ -5 \end{bmatrix} \text{ is an eigenvector.}$$

$$(ii)\ \begin{bmatrix} 1 & 6 \\ 5 & 2 \end{bmatrix}\begin{bmatrix} 3 \\ -2 \end{bmatrix} = \begin{bmatrix} -9 \\ 11 \end{bmatrix} \neq \lambda\begin{bmatrix} 3 \\ -2 \end{bmatrix} \Rightarrow \begin{bmatrix} 3 \\ -2 \end{bmatrix} \text{ is not an eigenvector.}$$

Ex.3.

Is 7 an eigenvalue of $\begin{bmatrix} 1 & 6 \\ 5 & 2 \end{bmatrix}$?

Answer.

7 is an eigenvalue $\Leftrightarrow$ $\boldsymbol{Ax} = 7\boldsymbol{x}$.

$\Leftrightarrow$ $\boldsymbol{Ax} - 7\boldsymbol{x} = \mathbf{0}$ $\Leftrightarrow$ $(\boldsymbol{A} - 7\mathrm{I})\boldsymbol{x} = \mathbf{0}$

$$\Leftrightarrow \left(\begin{bmatrix} 1 & 6 \\ 5 & 2 \end{bmatrix} - \begin{bmatrix} 7 & 0 \\ 0 & 7 \end{bmatrix}\right)\boldsymbol{x} = 0$$

$$\Leftrightarrow \begin{bmatrix} -6 & 6 \\ 5 & -5 \end{bmatrix}\boldsymbol{x} = 0 \quad \equiv \begin{bmatrix} 1 & -1 \\ 0 & 0 \end{bmatrix}\boldsymbol{x} = 0$$

$$\Leftrightarrow \boldsymbol{x} = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$$

$\Leftrightarrow$ 7 is an eigenvalue.

$$\boldsymbol{A} = \begin{bmatrix} 1 & 6 \\ 5 & 2 \end{bmatrix} \text{ and } \boldsymbol{x} = \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}$$

$$\boldsymbol{Ax} - 7\boldsymbol{x} = \begin{bmatrix} x_1 + 6x_2 \\ 5x_1 + 2x_2 \end{bmatrix} - \begin{bmatrix} 7x_1 \\ 7x_2 \end{bmatrix} = \begin{bmatrix} -6x_1 + 6x_2 \\ 5x_1 - 5x_2 \end{bmatrix}$$

$$(\boldsymbol{A} - 7\mathbf{I})\boldsymbol{x} = \begin{bmatrix} 1-7 & 6 \\ 5 & 2-7 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} -6x_1 + 6x_2 \\ 5x_1 - 5x_2 \end{bmatrix}$$

Conclusion

(*i*) λ is an eigenvalue of $\boldsymbol{A} \Leftrightarrow (\boldsymbol{A}-\lambda \mathrm{I})\boldsymbol{x} = \mathbf{0}$ has a nontrivial solution.

(*ii*) The set of all solutions of $(\boldsymbol{A}-\lambda \mathrm{I})\boldsymbol{x} = \mathbf{0}$ is just the Nul $(\boldsymbol{A}-\lambda \mathrm{I})$. So this set is a subspace of $\boldsymbol{R}^n$ and is called the eigenspace of $\boldsymbol{A}$ corresponding to λ. In other words, the spanning set of all eigenvectors of $\boldsymbol{A}$ is called the eigenspace of $\boldsymbol{A}$ corresponding to λ.

(Note: a distinct eigenvalue generates one eigenspace.)

Ex.4.

Find the eigenspace of $\begin{bmatrix} 4 & -1 & 6 \\ 2 & 1 & 6 \\ 2 & -1 & 8 \end{bmatrix}$ corresponding to 2.

Answer.

$$\text{To find Nul}\left(\begin{bmatrix} 4 & -1 & 6 \\ 2 & 1 & 6 \\ 2 & -1 & 8 \end{bmatrix} - \begin{bmatrix} 2 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 2 \end{bmatrix}\right) = \text{Nul}\left(\begin{bmatrix} 2 & -1 & 6 \\ 2 & -1 & 6 \\ 2 & -1 & 6 \end{bmatrix}\right)$$

$$\begin{bmatrix} 2 & -1 & 6 \\ 2 & -1 & 6 \\ 2 & -1 & 6 \end{bmatrix} \xrightarrow{\text{row reduce}} \begin{bmatrix} 2 & -1 & 6 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

$$\Rightarrow \ 2x_1 - x_2 + 6x_3 = 0$$

$$\Rightarrow \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} \tfrac{1}{2}x_2 - 3x_3 \\ x_2 \\ x_3 \end{bmatrix} = x_2\begin{bmatrix} \tfrac{1}{2} \\ 1 \\ 0 \end{bmatrix} + x_3\begin{bmatrix} -3 \\ 0 \\ 1 \end{bmatrix}.$$

$$\text{A basis for Nul } \boldsymbol{A} \text{ is } \left\{\begin{bmatrix} 1 \\ 2 \\ 0 \end{bmatrix}, \begin{bmatrix} -3 \\ 0 \\ 1 \end{bmatrix}\right\}.$$

A special case of eigenvalue

- Theorem 1
  Let $\boldsymbol{A}$ be a triangular matrix. Then the eigenvalues of $\boldsymbol{A}$ are the entries on its main diagonal.

- Ex.5.
  The eigenvalues of $\begin{bmatrix} 3 & 6 & -8 \\ 0 & 0 & 6 \\ 0 & 0 & 2 \end{bmatrix}$ are 3, 0, 2; since

$$\left|\begin{bmatrix} 3-\lambda & 6 & -8 \\ 0 & 0-\lambda & 6 \\ 0 & 0 & 2-\lambda \end{bmatrix}\right| = (3-\lambda)(-\lambda)(2-\lambda) = 0$$

$$\Rightarrow \lambda_1 = 3,\ \lambda_2 = 0, \lambda_3 = 2.$$

- Theorem 2
  If $\boldsymbol{v}_1, \ldots, \boldsymbol{v}_r$ are eigenvectors corresponding to distinct eigenvalues $\lambda_1, \ldots, \lambda_r$ of $\boldsymbol{A}_{n\times n}$, then $\{\boldsymbol{v}_1, \ldots, \boldsymbol{v}_r\}$ is linearly independent.

  proof.
  Take $\{\boldsymbol{v}_i\}$ being linearly dependent to derive a contradiction.
  If $\{\boldsymbol{v}_1, \ldots, \boldsymbol{v}_r\}$ is linearly dependent, then there is a least index p such that $\boldsymbol{v}_{p+1}$ is a linear combination of the preceding (linearly independent) vectors, and there exist scalars $c_1, \ldots, c_p$ such that

$$c_1\boldsymbol{v}_1 + \ldots + c_p\boldsymbol{v}_p = \boldsymbol{v}_{p+1} \quad \ldots\ldots\ldots\ldots \quad (1)$$

  Multiplying both sides of the equation by $\boldsymbol{A}$,

$$c_1\boldsymbol{A}\boldsymbol{v}_1 + \ldots + c_p\boldsymbol{A}\boldsymbol{v}_p = \boldsymbol{A}\boldsymbol{v}_{p+1}.$$

Since $\boldsymbol{Av}_k = \lambda_k \boldsymbol{v}_k$ for each k, we have

$$c_1 \lambda_1 \boldsymbol{v}_1 + \ldots + c_p \lambda_p \boldsymbol{v}_p = \lambda_{p+1} \boldsymbol{v}_{p+1} \quad \text{.......................... (2)}$$

Multiplying both sides of Eq.(1) by λ_{p+1}

$$c_1 \lambda_{p+1} \boldsymbol{v}_1 + \ldots + c_p \lambda_{p+1} \boldsymbol{v}_p = \lambda_{p+1} \boldsymbol{v}_{p+1} \quad \text{.................... (3)}$$

Eq.(2) - Eq.(3)

$$\Rightarrow c_1 (\lambda_1 - \lambda_{p+1}) \boldsymbol{v}_1 + \ldots + c_p (\lambda_p - \lambda_{p+1}) \boldsymbol{v}_p = 0$$

Since $\{\boldsymbol{v}_1, \ldots, \boldsymbol{v}_p\}$ is linearly independent,

$$c_1 (\lambda_1 - \lambda_{p+1}) = c_2 (\lambda_2 - \lambda_{p+1}) = \ldots = c_p (\lambda_p - \lambda_{p+1}) = 0$$

but $\lambda_i \neq \lambda_j$ for $i \neq j$

Thus $c_1 = c_2 = \ldots = c_p = 0$

$$\Rightarrow \boldsymbol{v}_{p+1} = 0$$

It is impossible since $\boldsymbol{v}_{p+1}$ is an eigenvector, thus $\{\boldsymbol{v}_1, \ldots, \boldsymbol{v}_r\}$ is linearly independent.

- Eigenvectors and difference equations

What is the solution of the difference equation

$$\boldsymbol{x}_{k+1} = \boldsymbol{A}\boldsymbol{x}_k,\ k = 0, 1, 2, ..$$

If $\boldsymbol{A}$ is $n \times n$ matrix, then the simplest solution of the difference equation is

$$\boldsymbol{x}_k = \lambda^k \boldsymbol{x}_0,\ k = 1, 2, ..$$

Where $\boldsymbol{x}_0$, is an eigenvector of matrix $\boldsymbol{A}$ corresponding to eigenvalue λ.

The derivation is described as follows

$$\boldsymbol{A}\boldsymbol{x}_k = \boldsymbol{A}(\lambda^k \boldsymbol{x}_0) = \lambda^k (\boldsymbol{A}\boldsymbol{x}_0) = \lambda^k (\lambda \boldsymbol{x}_0) = \lambda^{k+1} \boldsymbol{x}_0 = \boldsymbol{x}_{k+1}.$$

- Exercises of Section 5.1.

5.2 The characteristic equation

- Ex.1.
  Find the eigenvalues of $\boldsymbol{A} = \begin{bmatrix} 2 & 3 \\ 3 & -6 \end{bmatrix}$.

  Solution.
  λ is an eigenvalue $\Leftrightarrow$ $(\boldsymbol{A} - \lambda\mathrm{I})\boldsymbol{x} = 0$ has a nontrivial solution
  $\Leftrightarrow$ $(\boldsymbol{A} - \lambda\mathrm{I})$ is not invertible $\Leftrightarrow$ $|\boldsymbol{A} - \lambda\mathrm{I}| = 0$
  Thus the eigenvalue λ satisfies the equation

  $|\boldsymbol{A} - \lambda\mathrm{I}| = 0 \equiv \begin{vmatrix} 2-\lambda & 3 \\ 3 & -6-\lambda \end{vmatrix} = 0$

  $\equiv -12 + 4\lambda + \lambda^2 - 9 = 0$
  $\equiv \lambda^2 + 4\lambda - 21 = 0$
  $\equiv (\lambda - 3)(\lambda + 7) = 0$
  $\equiv \lambda = 3$ or -7.

- Conclusion
  A scalar λ is an eigenvalue of an $n \times n$ matrix $\boldsymbol{A}$
  $\Leftrightarrow$ λ satisfies the characteristic equation $\det(\boldsymbol{A} - \lambda\mathrm{I}) = 0$.

- Note
  If $\boldsymbol{A}$ is an $n \times n$ matrix, then $\det(\boldsymbol{A} - \lambda\,\mathrm{I})$ is a polynomial of degree n called the characteristic polynomial of $\boldsymbol{A}$, and there are n roots for the polynomial.

- Ex.4.
  $\lambda^6 - 4\lambda^5 - 12\lambda^4$ is the characteristic polynomial of a 6×6 matrix $\boldsymbol{A}$.
  $\lambda^4(\lambda^2 - 4\lambda - 12) = 0$
  $\Rightarrow \lambda^4(\lambda - 6)(\lambda + 2) = 0$
  $\Rightarrow \lambda = 0, 0, 0, 0, 6, -2$.

- Theorem (The invertible matrix theorem)
  Let $\boldsymbol{A}$ be a $n \times n$ matrix. Then $\boldsymbol{A}$ is invertible if and only if
  s. The number 0 is not an eigenvalue of $\boldsymbol{A}$.
  t. The determinant of $\boldsymbol{A}$ is not zero.

  Proof of *s*.
  0 is an eigenvalue of $\boldsymbol{A}$ $\Leftrightarrow$ $\boldsymbol{Ax} - 0\boldsymbol{x} = \mathbf{0}$
  $\Leftrightarrow$ $\boldsymbol{Ax} = \mathbf{0}$ has nontrivial solution
  $\Leftrightarrow$ $\boldsymbol{A}$ is not invertible.

- Note that
  i. **0** can not be an eigenvector.
  ii. 0 may be an eigenvalue.

Similarity

- Definition
  $\boldsymbol{A}_{n\times n}$ is similar $\boldsymbol{B}_{n\times n}$ if there is an invertible matrix $\boldsymbol{P}$ such that $\boldsymbol{P}^{-1}\boldsymbol{A}\boldsymbol{P} = \boldsymbol{B}$ or equivalently, $\boldsymbol{A} = \boldsymbol{P}\boldsymbol{B}\boldsymbol{P}^{-1}$.

- Theorem 4
  If $n \times n$ matrix $\boldsymbol{A}$ and $\boldsymbol{B}$ are similar, then they have the same characteristic polynomial and hence the same eigenvalues.
  Proof.
  If $\boldsymbol{B} = \boldsymbol{P}^{-1}\boldsymbol{AP}$, then
  $$\boldsymbol{B} - \lambda \mathrm{I} = \boldsymbol{P}^{-1}\boldsymbol{AP} - \lambda\, \boldsymbol{P}^{-1}\boldsymbol{P} = \boldsymbol{P}^{-1}(\boldsymbol{AP} - \lambda \boldsymbol{P}) = \boldsymbol{P}^{-1}(\boldsymbol{A} - \lambda \mathrm{I})\, \boldsymbol{P}$$
  $\Rightarrow$ $\det(\boldsymbol{B} - \lambda \mathrm{I}) = \det(\boldsymbol{P}^{-1}) \det(\boldsymbol{A} - \lambda \mathrm{I}) \det(\boldsymbol{P})$
  $\Rightarrow$ $\det(\boldsymbol{B} - \lambda \mathrm{I}) = \det(\boldsymbol{A} - \lambda \mathrm{I}) \det(\boldsymbol{P}^{-1}) \det(\boldsymbol{P}) = \det(\boldsymbol{A} - \lambda \mathrm{I})$.

Summation and multiplication of all eigenvalues for matrix A

- For any $n \times n$ square matrix $\boldsymbol{A}$, $\begin{bmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ a_{n1} & a_{n2} & \cdots & a_{nn} \end{bmatrix}$

  The *trace* of $\boldsymbol{A}_{n \times n}$ is the sum of the diagonal entries in $\boldsymbol{A}$ and is denoted by $tr\,\boldsymbol{A}$.

- If $\lambda_1, \lambda_2, .., \lambda_n$ are the eigenvalues of $\boldsymbol{A}$.

  The sum of all eigenvalues of $\boldsymbol{A}$ is equal to the trace of $\boldsymbol{A}$.

  $$\lambda_1 + \lambda_2 + .. + \lambda_n = a_{11} + a_{22} + .. + a_{nn} = tr\,\boldsymbol{A}.$$

  The multiplication of all eigenvalues of $\boldsymbol{A}$ is equal to the determinant of $\boldsymbol{A}$.

  $$\lambda_1 \lambda_2 .. \lambda_n = det\,\boldsymbol{A}.$$

1. For 2×2 matrix $\boldsymbol{A}$, $\begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{bmatrix}$

   $det(\boldsymbol{A} - \lambda \boldsymbol{I}) = \lambda^2 - (a_{11} + a_{22})\lambda + (a_{11}a_{22} - a_{12}a_{21})$

   $(\lambda - \lambda_1)(\lambda - \lambda_2) = \lambda^2 - (\lambda_1 + \lambda_2)\lambda + (\lambda_1\lambda_2)$

   Thus $a_{11} + a_{22} = \lambda_1 + \lambda_2$

   $det(\boldsymbol{A} - \lambda \boldsymbol{I}) = \lambda_1\lambda_2$

2. For 3×3 matrix $\boldsymbol{A}$, $\begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{bmatrix}$

   $det(\boldsymbol{A} - \lambda \boldsymbol{I}) = \lambda^3 - (a_{11} + a_{22} + a_{33})\lambda^2 + .. + det\,\boldsymbol{A}$

   $(\lambda - \lambda_1)(\lambda - \lambda_2)(\lambda - \lambda_3) = \lambda^3 - (\lambda_1 + \lambda_2 + \lambda_3)\lambda + .. + (\lambda_1\lambda_2\lambda_3)$

   Thus $a_{11} + a_{22} + a_{33} = \lambda_1 + \lambda_2 + \lambda_3$

   $det(\boldsymbol{A} - \lambda \boldsymbol{I}) = \lambda_1\lambda_2\lambda_3$

3. For 4×4 matrix ***A***,

$det(\boldsymbol{A} - \lambda \boldsymbol{I}) = \lambda^4 - (a_{11} + a_{22} + a_{33} + a_{44})\,\lambda^3 + .. + det\,\boldsymbol{A}$

$(\lambda - \lambda_1)(\lambda - \lambda_2)(\lambda - \lambda_3)(\lambda - \lambda_4) = \lambda^4 - (\lambda_1 + \lambda_2 + \lambda_3 + \lambda_4)\,\lambda^3 + .. + (\lambda_1 \lambda_2 \lambda_3 \lambda_4)$

Thus $a_{11} + a_{22} + a_{33} + a_{44} = \lambda_1 + \lambda_2 + \lambda_3 + \lambda_4$

$det(\boldsymbol{A} - \lambda \boldsymbol{I}) = \lambda_1 \lambda_2 \lambda_3 \lambda_4$

4. For $n \times n$ matrix ***A***,

$det(\boldsymbol{A} - \lambda \boldsymbol{I}) = \lambda^n - (a_{11} + a_{22} + .. + a_{nn})\,\lambda^{n-1} + .. + det\,\boldsymbol{A}$

$(\lambda - \lambda_1)(\lambda - \lambda_2)\,..\,(\lambda - \lambda_n) = \lambda^n - (\lambda_1 + \lambda_2 + .. + \lambda_n)\,\lambda^{n-1} + .. + (\lambda_1 \lambda_2\,..\,\lambda_n)$

Thus $a_{11} + a_{22} + .. + a_{nn} = \lambda_1 + \lambda_2 + .. + \lambda_n$

$det(\boldsymbol{A} - \lambda \boldsymbol{I}) = \lambda_1 \lambda_2\,..\,\lambda_n$

- Exercises of Section 5.2.

5.3 Diagonalization

- A square matrix ***A*** is said to be diagonalizable if ***A*** is similar to a diagonal matrix; that is, if $\boldsymbol{A} = \boldsymbol{P}\boldsymbol{D}\boldsymbol{P}^{-1}$ for some invertible matrix ***P*** and some diagonal matrix ***D***.

- Theorem 5 (The Diagonalization Theorem)

  (*i*) An $n \times n$ matrix ***A*** is diagonalizable if and only if ***A*** has *n* linearly independent eigenvectors.

  (*ii*) If $\boldsymbol{A} = \boldsymbol{P}\boldsymbol{D}\boldsymbol{P}^{-1}$ where ***D*** is diagonal, then the diagonal entries of ***D*** are eigenvalues of ***A*** and the columns of ***P*** are corresponding eigenvectors.

  Proof.

  If $\boldsymbol{P}_{n\times n} = [\boldsymbol{u}_1, \ldots, \boldsymbol{u}_n]$ and $\boldsymbol{D} = \begin{bmatrix} \lambda_1 & & \\ & \ddots & \\ & & \lambda_n \end{bmatrix}$, then

  $\boldsymbol{AP} = \boldsymbol{A}[\boldsymbol{u}_1, \ldots, \boldsymbol{u}_n] = [\boldsymbol{A}\boldsymbol{u}_1, \ldots, \boldsymbol{A}\boldsymbol{u}_n]$,

  while $\boldsymbol{P}\boldsymbol{D} = [\lambda_1\boldsymbol{u}_1, \ldots, \lambda_n\boldsymbol{u}_n]$

"⇒" Suppose that $\boldsymbol{A}$ is diagonalizable and $\boldsymbol{A} = \boldsymbol{PDP}^{-1}$.
Then $\boldsymbol{AP} = \boldsymbol{PDP}^{-1}\boldsymbol{P} = \boldsymbol{PD}$
$\Rightarrow [\boldsymbol{Au}_1, \ldots, \boldsymbol{Au}_n] = [\lambda_1\boldsymbol{u}_1, \ldots, \lambda_n\boldsymbol{u}_n]$
$\Rightarrow \boldsymbol{Au}_i = \lambda_i \boldsymbol{u}_i$ for $i = 1, 2, \ldots, n$.
That is $\boldsymbol{u}_i$'s are eigenvectors of $\boldsymbol{A}$.
Since $\boldsymbol{P}$ is invertible, its columns $\boldsymbol{u}_1, \ldots, \boldsymbol{u}_n$ must be linearly independent .

"⇐" Just take linearly independent vectors $\boldsymbol{u}_1, \ldots, \boldsymbol{u}_n$ to construct the matrix $\boldsymbol{P} = [\boldsymbol{u}_1, \ldots, \boldsymbol{u}_n]$ and the corresponding eigenvalues to

construct $\boldsymbol{D} = \begin{bmatrix} \lambda_1 & & \\ & \ddots & \\ & & \lambda_n \end{bmatrix}$.

Then $\boldsymbol{AP} = \boldsymbol{PD}$. Since $\boldsymbol{u}_i$'s are linearly independent, $\boldsymbol{P}$ is invertible, $\boldsymbol{A} = \boldsymbol{PDP}^{-1}$. Thus $\boldsymbol{A}$ is similar to $\boldsymbol{D}$, $\boldsymbol{A}$ is diagonalizable.

- Note
  $\boldsymbol{A}_{n\times n}$ is diagonalizable ⇔ there are n eigenvectors to form a basis for $\boldsymbol{R}^n$; We call the basis an eigenvector basis.

Diagonalizing matrices

- Ex.3.
  Diagonalize matrix $\boldsymbol{A} = \begin{bmatrix} 1 & 3 & 3 \\ -3 & -5 & -3 \\ 3 & 3 & 1 \end{bmatrix}$. That is, find an invertible matrix $\boldsymbol{P}$ and a diagonal matrix $\boldsymbol{D}$ such that $\boldsymbol{A} = \boldsymbol{PDP}^{-1}$.
  Solution.
  Four steps:
  *step*1. Find the eigenvalues of $\boldsymbol{A}$. det $(\boldsymbol{A} - \lambda \boldsymbol{I}) = \begin{vmatrix} 1-\lambda & 3 & 3 \\ -3 & -5-\lambda & -3 \\ 3 & 3 & 1-\lambda \end{vmatrix}$

$$= -\lambda^3 - 3\lambda^2 + 4 = -(\lambda - 1)(\lambda + 2)^2 \Rightarrow \lambda = 1, -2, -2 .$$

step 2 . Find three linearly independent eigenvectors of ***A***

$$(\boldsymbol{A}-1\boldsymbol{I})\boldsymbol{x}=0 \Rightarrow \begin{bmatrix} 0 & 3 & 3 \\ -3 & -6 & -3 \\ 3 & 3 & 0 \end{bmatrix}\boldsymbol{x}=0 \Rightarrow \begin{bmatrix} 1 & 1 & 0 \\ 0 & 1 & 1 \\ 0 & 0 & 0 \end{bmatrix}\boldsymbol{x}=0$$

$$\Rightarrow \boldsymbol{x}=\begin{bmatrix} 1 \\ -1 \\ 1 \end{bmatrix}$$

$$(\boldsymbol{A}-(-2)\boldsymbol{I})\boldsymbol{x}=0 \Rightarrow \begin{bmatrix} 3 & 3 & 3 \\ -3 & -3 & -3 \\ 3 & 3 & 3 \end{bmatrix}\boldsymbol{x}=0 \Rightarrow \begin{bmatrix} 1 & 1 & 1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}\boldsymbol{x}=0$$

$$\Rightarrow \boldsymbol{x}=\begin{bmatrix} 1 \\ -1 \\ 0 \end{bmatrix}, \begin{bmatrix} 1 \\ 0 \\ -1 \end{bmatrix}.$$

step 3. Construct ***D*** from the eigenvalues, $\boldsymbol{D}=\begin{bmatrix} 1 & 0 & 0 \\ 0 & -2 & 0 \\ 0 & 0 & -2 \end{bmatrix}.$

step 4. Construct ***P*** from the eigenvectors according to the order of eigenvalues,

$$\boldsymbol{P}=\begin{bmatrix} 1 & 1 & 1 \\ -1 & -1 & 0 \\ 1 & 0 & -1 \end{bmatrix}.$$

Check the solution.

$$\boldsymbol{AP}=\begin{bmatrix} 1 & 3 & 3 \\ -3 & -5 & -3 \\ 3 & 3 & 1 \end{bmatrix}\begin{bmatrix} 1 & 1 & 1 \\ -1 & -1 & 0 \\ 1 & 0 & -1 \end{bmatrix}=\begin{bmatrix} 1 & -2 & -2 \\ -1 & 2 & 0 \\ 1 & 0 & 2 \end{bmatrix}.$$

$$\boldsymbol{PD}=\begin{bmatrix} 1 & 1 & 1 \\ -1 & -1 & 0 \\ 1 & 0 & -1 \end{bmatrix}\begin{bmatrix} 1 & 0 & 0 \\ 0 & -2 & 0 \\ 0 & 0 & -2 \end{bmatrix}=\begin{bmatrix} 1 & -2 & -2 \\ -1 & 2 & 0 \\ 1 & 0 & 2 \end{bmatrix}.$$

- Ex.4.
  Diagonalize the matrix $A = \begin{bmatrix} 2 & 4 & 3 \\ -4 & -6 & -3 \\ 3 & 3 & 1 \end{bmatrix}$, if possible.

  $\det(A - \lambda I) = 0 \Rightarrow -(\lambda - 1)(\lambda + 2)^2.$

  $(A - 1I)x = 0 \Rightarrow x = \begin{bmatrix} 1 \\ -1 \\ 1 \end{bmatrix}.$

  $(A - (-2)I)x = 0 \Rightarrow \begin{bmatrix} 4 & 4 & 3 \\ -4 & -4 & -3 \\ 3 & 3 & 3 \end{bmatrix} x = 0 \Rightarrow \begin{bmatrix} 1 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{bmatrix} x = 0$

  $\Rightarrow x = \begin{bmatrix} 1 \\ -1 \\ 0 \end{bmatrix}$ just one linearly independent vector.

  There are no three linearly independent eigenvectors.
  Thus ***A*** is not diagonalizable.

- Theorem 6
  If an $n \times n$ matrix ***A*** has n distinct eigenvalues, then ***A*** is diagonalizable.

  Proof.
  A has n distinct eigenvalues
  $\Rightarrow$ ***A*** has n linearly independent eigenvectors
  $\Rightarrow$ ***A*** is diagonalizable.

- Note
  Distinct eigenvalues $\Rightarrow$ diagonalizable
  but diagonalizable $\nRightarrow$ distinct eigenvalues; for example, Ex.3.

Matrices where eigenvalues are not distinct

- Theorem 7

$\boldsymbol{A}_{n\times n}$ has p distinct eigenvalues $\lambda_1, \ldots, \lambda_p$. For $k = 1, \ldots, p$, let β_k be a basis for the eigenspace corresponding to λ_k. Let

$$\beta = \beta_1 \cup \beta_2 \cup \ldots \cup \beta_p .$$

Then

i. β is a linearly independent set of vectors in $\boldsymbol{R}^n$.

ii. $\boldsymbol{A}$ is diagonalizable if and only if β contains n vectors.

Power of matrices

Example. If $\boldsymbol{D} = \begin{bmatrix} 5 & 0 \\ 0 & 3 \end{bmatrix}$, then $\boldsymbol{D}^k = \begin{bmatrix} 5^k & 0 \\ 0 & 3^k \end{bmatrix}$ for $k \geq 1$.

If $\boldsymbol{A} = \begin{bmatrix} 7 & 2 \\ -4 & 1 \end{bmatrix}$, then what is $\boldsymbol{A}^k$?

Answer. $\boldsymbol{A} = \boldsymbol{PDP}^{-1} \Rightarrow \boldsymbol{A}^2 = \boldsymbol{PDP}^{-1}\boldsymbol{PDP}^{-1} = \boldsymbol{PD}^2\boldsymbol{P}^{-1} \Rightarrow$
$\boldsymbol{A}^3 = \boldsymbol{PDP}^{-1}\boldsymbol{PD}^2\boldsymbol{P}^{-1} = \boldsymbol{PD}^3\boldsymbol{P}^{-1} \Rightarrow \ldots \Rightarrow \boldsymbol{A}^k = \boldsymbol{PD}^k\boldsymbol{P}^{-1}$

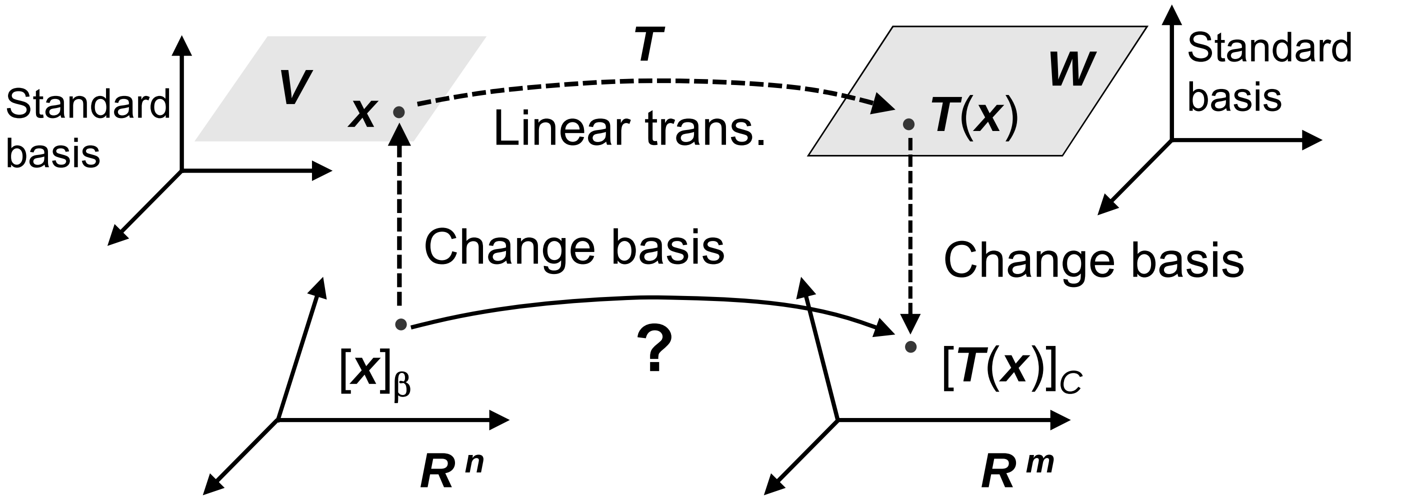
$$\boldsymbol{A} = \begin{bmatrix} 1 & 1 \\ -1 & -2 \end{bmatrix}\begin{bmatrix} 5 & 0 \\ 0 & 3 \end{bmatrix}\begin{bmatrix} 2 & 1 \\ -1 & -1 \end{bmatrix}$$

$$\boldsymbol{A}^k = \begin{bmatrix} 1 & 1 \\ -1 & -2 \end{bmatrix}\begin{bmatrix} 5^k & 0 \\ 0 & 3^k \end{bmatrix}\begin{bmatrix} 2 & 1 \\ -1 & -1 \end{bmatrix}$$

- Exercises of Section 5.3.

5.4 Eigenvectors and linear transformations

- Goal
  To understand the matrix factorization $\boldsymbol{A} = \boldsymbol{P} \boldsymbol{D} \boldsymbol{P}^{-1}$ as a statement about linear transformations. We know that a linear transformation maps vectors from $\boldsymbol{R}^n$ to $\boldsymbol{R}^m$ in standard bases, but changing basis means to map vectors from $\boldsymbol{R}^n$ to $\boldsymbol{R}^n$ in different bases. What is the standard matrix to transform vectors from $\boldsymbol{R}^n$ to $\boldsymbol{R}^m$ in different bases ?



- The connection between $[\boldsymbol{x}]_\beta$ and $[\boldsymbol{T}(\boldsymbol{x})]_C$ is easy to find. Let $\{\boldsymbol{b}_1, \boldsymbol{b}_2, , \boldsymbol{b}_n\}$ be the basis β for $\boldsymbol{V}$. If $\boldsymbol{x} = r_1\boldsymbol{b}_1 + r_2b_2 + \ldots + r_nb_n$,

  then $[\boldsymbol{x}]_{\boldsymbol{\beta}} = \begin{bmatrix} r_1 \\ : \\ r_n \end{bmatrix}$

  and $T(\boldsymbol{x}) = T(r_1\boldsymbol{b}_1 + .. + r_n\boldsymbol{b}_n) = r_1T(\boldsymbol{b}_1) + .. + r_nT(\boldsymbol{b}_n)$.

  Using the basis C in $\boldsymbol{W}$, we can rewrite the equation in terms of C-coordinate vectors:

  $$[\boldsymbol{T}(\boldsymbol{x})]_C = r_1[\boldsymbol{T}(\boldsymbol{b}_1)]_C + r_2[\boldsymbol{T}(b_2)]_C + \ldots + r_n[\boldsymbol{T}(b_n)]_C.$$

  Since C-coordinate vectors are in $\boldsymbol{R}^m$, this vector can be written as a matrix equation, namely,

  $$[\boldsymbol{T}(\boldsymbol{x})]_C = \boldsymbol{M}\,[\boldsymbol{x}]_\beta$$

  where $\boldsymbol{M} = [\ [\boldsymbol{T}(\boldsymbol{b}_1)]_C\ \ [\boldsymbol{T}(b_2)]_C\ \ \ldots\ [\boldsymbol{T}(b_n)]_C]$.
  The matrix $\boldsymbol{M}$ is a matrix representation of $\boldsymbol{T}$, called the matrix for $\boldsymbol{T}$ relative to the bases β and C.

- Ex.1.

  Suppose $\beta = \{\boldsymbol{b}_1, \boldsymbol{b}_2\}$ is a basis for $\boldsymbol{V}$ and $C = \{\boldsymbol{c}_1, \boldsymbol{c}_2, \boldsymbol{c}_3\}$ is a basis for $\boldsymbol{W}$. Let $\boldsymbol{T}: \boldsymbol{V} \to \boldsymbol{W}$ be a linear transformation with the property that

  $$\boldsymbol{T}(\boldsymbol{b}_1) = 3\boldsymbol{c}_1 - 2\boldsymbol{c}_2 + 5\boldsymbol{c}_3 \quad \text{and} \quad \boldsymbol{T}(b_2) = 4\boldsymbol{c}_1 + 7\boldsymbol{c}_2 - \boldsymbol{c}_3.$$

  Find the matrix $\boldsymbol{M}$ for $\boldsymbol{T}$ relative to β and C.

  Solution.

  The C-coordinate vectors of the images of $\boldsymbol{b}_1$ and $\boldsymbol{b}_2$ are

  $$[T(\boldsymbol{b}_1)]_C = \begin{bmatrix} 3 \\ -2 \\ 5 \end{bmatrix} \quad \text{and} \quad [T(\boldsymbol{b}_2)]_C = \begin{bmatrix} 4 \\ 7 \\ -1 \end{bmatrix}$$

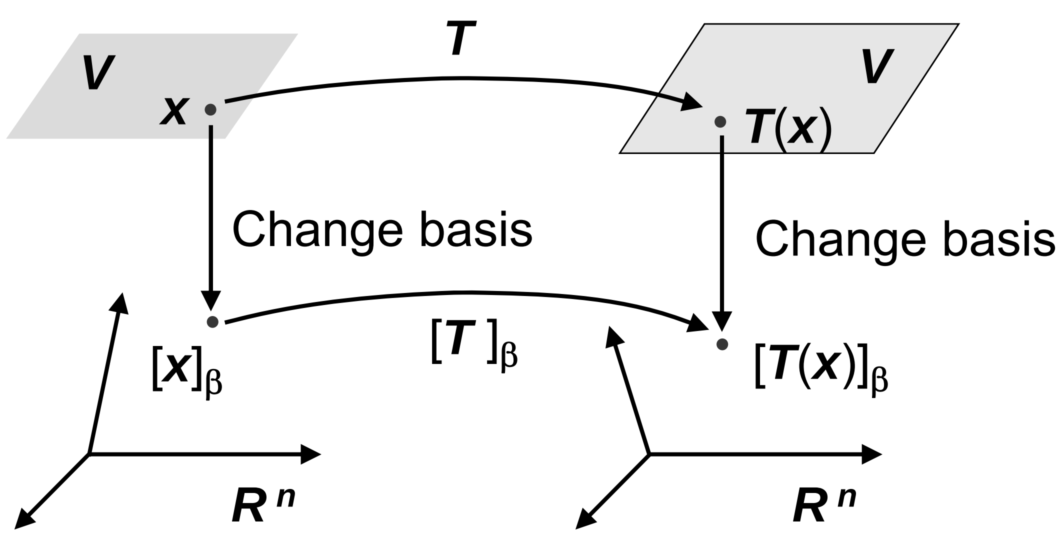
  Hence

  $$\boldsymbol{M} = \begin{bmatrix} 3 & 4 \\ -2 & 7 \\ 5 & -1 \end{bmatrix}.$$

Linear transformations from V to V

- In the common case when $\boldsymbol{W}$ is the same as $\boldsymbol{V}$ and the basis C is the same as β, the matrix is called the matrix for $\boldsymbol{T}$ relative to β, or simply the β-matrix for $\boldsymbol{T}$, and is denoted by $[\boldsymbol{T}]_\beta$. The β-matrix for $\boldsymbol{T}: \boldsymbol{V} \to \boldsymbol{V}$ satisfies

  $$[\boldsymbol{T}(\boldsymbol{x})]_\beta = [\boldsymbol{T}]_\beta\, [\boldsymbol{x}]_\beta, \text{ for all } \boldsymbol{x} \text{ in } \boldsymbol{V}.$$



Linear transformations on $\boldsymbol{R}^n$

- Theorem 8 (Diagonal matrix representation)
  Suppose $\boldsymbol{A} = \boldsymbol{P}\boldsymbol{D}\boldsymbol{P}^{-1}$, where $\boldsymbol{D}$ is a diagonal $n \times n$ matrix. If β is the basis for $\boldsymbol{R}^n$ formed from the columns of $\boldsymbol{P}$, then $\boldsymbol{D}$ is the β-matrix for the transformation $\boldsymbol{x} \mapsto \boldsymbol{Ax}$.

  Proof.
  Denote the columns of $\boldsymbol{P}$ by $\boldsymbol{b}_1, .., \boldsymbol{b}_n$, so that $\beta = \{\boldsymbol{b}_1, .., \boldsymbol{b}_n\}$ and $\boldsymbol{P} = [\boldsymbol{b}_1, .., \boldsymbol{b}_n]$. In this case, $\boldsymbol{P}$ is the change-of-coordinates matrix $\boldsymbol{P}_\beta$, where $\boldsymbol{P}[\boldsymbol{x}]_\beta = \boldsymbol{x}$ and $[\boldsymbol{x}]_\beta = \boldsymbol{P}^{-1}\boldsymbol{x}$.
  If $\boldsymbol{T}(\boldsymbol{x}) = \boldsymbol{Ax}$ for $\boldsymbol{x}$ in $\boldsymbol{R}^n$, then

  $$[\boldsymbol{T}]_\beta = [\,[\boldsymbol{T}(\boldsymbol{b}_1)]_\beta \dots [\boldsymbol{T}(\boldsymbol{b}_n)]_\beta\,] \qquad \text{(refer to page 28)}$$
  $$= [\,[\boldsymbol{A}\boldsymbol{b}_1]_\beta \dots [\boldsymbol{A}\boldsymbol{b}_n]_\beta\,]$$
  $$= [\,\boldsymbol{P}^{-1}\boldsymbol{A}\boldsymbol{b}_1 \dots \boldsymbol{P}^{-1}\boldsymbol{A}\boldsymbol{b}_n\,]$$
  $$= \boldsymbol{P}^{-1}\boldsymbol{A}\,[\boldsymbol{b}_1 \dots \boldsymbol{b}_n\,]$$
  $$= \boldsymbol{P}^{-1}\boldsymbol{A}\,\boldsymbol{P}$$

  Since $\boldsymbol{A} = \boldsymbol{P}\boldsymbol{D}\boldsymbol{P}^{-1}$, we have $[\boldsymbol{T}]_\beta = \boldsymbol{P}^{-1}\boldsymbol{A}\boldsymbol{P} = \boldsymbol{D}$.

- Ex. 3.
  Define $\boldsymbol{T}: \boldsymbol{R}^2 \to \boldsymbol{R}^2$ by $\boldsymbol{T}(\boldsymbol{x}) = \boldsymbol{Ax}$, where $\boldsymbol{A} = \begin{bmatrix} 7 & 2 \\ -4 & 1 \end{bmatrix}$.

  Find a basis β for $\boldsymbol{R}^2$ with the property that the β–matrix for $\boldsymbol{T}$ is a diagonal matrix.

  Solution.

  $\boldsymbol{A} = \boldsymbol{PDP}^{-1}$, where $\boldsymbol{P} = \begin{bmatrix} 1 & 1 \\ -1 & -2 \end{bmatrix}$ and $\boldsymbol{D} = \begin{bmatrix} 5 & 0 \\ 0 & 3 \end{bmatrix}$.

  The columns of $\boldsymbol{P}$, call them $\boldsymbol{b}_1$ and $\boldsymbol{b}_2$, are eigenvectors of $\boldsymbol{A}$. $\boldsymbol{D}$ is the β–matrix for $\boldsymbol{T}$ when $\beta = \{\boldsymbol{b}_1, \boldsymbol{b}_2\}$.

  The mappings $\boldsymbol{x} \mapsto \boldsymbol{Ax}$ and $\boldsymbol{u} \mapsto \boldsymbol{Du}$ describe the same linear transformation, relative to different bases.

Similarity of matrix representations

- The proof of Theorem 8 did not use the information that ***D*** was diagonal. Hence, if ***A*** is similar to a matrix ***C***, with $\boldsymbol{A} = \boldsymbol{P}\,\boldsymbol{C}\,\boldsymbol{P}^{-1}$, then ***C*** is the β-matrix for the transformation $\boldsymbol{x} \mapsto \boldsymbol{A}\boldsymbol{x}$ when the basis β is formed the columns of ***P***.
- Ex. 4. Let $\boldsymbol{A} = \begin{bmatrix} 4 & -9 \\ 4 & 8 \end{bmatrix}$, $\boldsymbol{b}_1 = \begin{bmatrix} 3 \\ 2 \end{bmatrix}$, and $\boldsymbol{b}_2 = \begin{bmatrix} 2 \\ 1 \end{bmatrix}$. Find the matrix.

  Solution.

  Though ***A*** is not diagonalization. However, the basis β = {$\boldsymbol{b}_1$, $\boldsymbol{b}_2$} has the property that the β -matrix for the transformation $\boldsymbol{x} \mapsto \boldsymbol{A}\boldsymbol{x}$ is a triangular matrix called the Jordan form of ***A***.

  If $\boldsymbol{P} = [\boldsymbol{b}_1, \boldsymbol{b}_2]$, then the β-matrix is $\boldsymbol{P}^{-1}\,\boldsymbol{A}\,\boldsymbol{P}$.

  $$\boldsymbol{P}^{-1}\boldsymbol{A}\boldsymbol{P} = \begin{bmatrix} -1 & 2 \\ 2 & -3 \end{bmatrix}\begin{bmatrix} 4 & -9 \\ 4 & -8 \end{bmatrix}\begin{bmatrix} 3 & 2 \\ 2 & 1 \end{bmatrix} = \begin{bmatrix} -2 & 1 \\ 0 & -2 \end{bmatrix}.$$

- Exercises of Section 5.4.

5.5 Complex eigenvalues

- The characteristic equation of an $n \times n$ matrix involves a polynomial of degree n, the equation always has exactly n roots including complex roots.
- Ex. 1. Matrix $\boldsymbol{A} = \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix}$ has complex eigenvalues.

  The characteristic equation is $\lambda^2 + 1 = 0$. Thus, ***A*** has eigenvalues i and $-i$. The corresponding eigenvectors are individually $\begin{bmatrix} 1 \\ -i \end{bmatrix}$ and $\begin{bmatrix} 1 \\ i \end{bmatrix}$.

- If a real polynomial equation has complex roots, the roots must be complex conjugate. That is, if a real square matrix has complex eigenvalues; the complex eigenvalues must be complex conjugate eigenvalues.

Real and imaginary parts of vectors

- The complex conjugate of a complex vector $\boldsymbol{x}$ in $\boldsymbol{C}^n$ is the vector $\overline{\boldsymbol{x}}$ in $\boldsymbol{C}^n$ whose entries are the complex conjugate of the entries in $\boldsymbol{x}$.
- Ex. 4. If $\boldsymbol{x} = \begin{bmatrix} 3-i \\ i \\ 2+5i \end{bmatrix} = \begin{bmatrix} 3 \\ 0 \\ 2 \end{bmatrix} + i \begin{bmatrix} -1 \\ 1 \\ 5 \end{bmatrix}$

  then $\overline{\boldsymbol{x}} = \begin{bmatrix} 3 \\ 0 \\ 2 \end{bmatrix} - i \begin{bmatrix} -1 \\ 1 \\ 5 \end{bmatrix} = \begin{bmatrix} 3+i \\ -i \\ 2-5i \end{bmatrix}$
- If $\boldsymbol{B}$ is an $m \times n$ matrix with possibly complex entries, then denotes the matrix whose entries are complex conjugate of the entries in $\boldsymbol{B}$.

Eigenvalues / eigenvectors of a real matrix that acts on $\boldsymbol{C}^n$

- Let $\boldsymbol{A}$ be an $n \times n$ matrix whose entries are real.

  Then $\overline{\boldsymbol{Ax}} = \overline{\boldsymbol{A}}\overline{\boldsymbol{x}} = \boldsymbol{A}\overline{\boldsymbol{x}}$.
- If λ is an eigenvalue of $\boldsymbol{A}$ and $\boldsymbol{x}$ is a corresponding eigenvector in $\boldsymbol{C}^n$, then $\boldsymbol{A}\overline{\boldsymbol{x}} = \overline{\boldsymbol{Ax}} = \overline{\lambda \boldsymbol{x}} = \overline{\lambda}\, \overline{\boldsymbol{x}}$.

  Hence $\overline{\lambda}$ is also an eigenvalue of A with $\overline{\boldsymbol{x}}$ a corresponding eigenvector. This shows that when $\boldsymbol{A}$ is real, its complex eigenvalues occur in conjugate pairs.
- Ex. 6. If $\boldsymbol{C} = \begin{bmatrix} a & -b \\ b & a \end{bmatrix}$, where a , b are real and not both zero,

  then the eigenvalues of $\boldsymbol{C}$ are $\lambda_1 = a+ib$, $\lambda_2 = a - ib$.

  The corresponding eigenvectors are $\begin{bmatrix} 1 \\ -i \end{bmatrix}$ and $\begin{bmatrix} 1 \\ i \end{bmatrix}$.

Theorem 9.

Let $\boldsymbol{A}$ be a real 2×2 matrix with a complex eigenvalue $\lambda = a - ib$ ($b \neq 0$) and an associated eigenvector $\boldsymbol{v}$ in $\boldsymbol{C}^2$.

Then $\boldsymbol{A} = \boldsymbol{P}\boldsymbol{C}\boldsymbol{P}^{-1}$, where $\boldsymbol{P} = [\text{Re}\,\boldsymbol{v}\ \text{Im}\,\boldsymbol{v}]$ and $\boldsymbol{C} = \begin{bmatrix} a & -b \\ b & a \end{bmatrix}$.

Proof.

Since λ is a complex eigenvalue of real matrix $\boldsymbol{A}$ and $\boldsymbol{v}$ is the corresponding eigenvector, $\boldsymbol{A}\boldsymbol{v} = \lambda\boldsymbol{v}$.

If $\boldsymbol{v} = \text{Re}\,\boldsymbol{v} + i\,\text{Im}\,\boldsymbol{v}$,

then $\boldsymbol{A}\boldsymbol{v} = \boldsymbol{A}(\text{Re}\,\boldsymbol{v}) + i\,\boldsymbol{A}(\text{Im}\,\boldsymbol{v})$

and
$$\begin{aligned}\lambda\boldsymbol{v} &= (a - ib)(\text{Re}\,\boldsymbol{v} + i\,\text{Im}\,\boldsymbol{v}) \\ &= a\,\text{Re}\,\boldsymbol{v} + i\,a\,\text{Im}\,\boldsymbol{v} - i\,b\,\text{Re}\,\boldsymbol{v} + b\,\text{Im}\,\boldsymbol{v} \\ &= (a\,\text{Re}\,\boldsymbol{v} + b\,\text{Im}\,\boldsymbol{v}) + i\,(-b\,\text{Re}\,\boldsymbol{v} + a\,\text{Im}\,\boldsymbol{v}).\end{aligned}$$

Thus
$$\begin{cases} \boldsymbol{A}(\text{Re}\,\boldsymbol{v}) = a\,\text{Re}\,\boldsymbol{v} + b\,\text{Im}\,\boldsymbol{v} \\ \boldsymbol{A}(\text{Im}\,\boldsymbol{v}) = -b\,\text{Re}\,\boldsymbol{v} + a\,\text{Im}\,\boldsymbol{v}. \end{cases}$$

Then $\boldsymbol{A}[\text{Re}\,\boldsymbol{v}\ \ \text{Im}\,\boldsymbol{v}] = [\text{Re}\,\boldsymbol{v}\ \ \text{Im}\,\boldsymbol{v}]\begin{bmatrix} a & -b \\ b & a \end{bmatrix}$.

That is $\boldsymbol{A}\boldsymbol{P} = \boldsymbol{P}\boldsymbol{C}$.

If we can prove that vectors $\text{Re}\,\boldsymbol{v}$ and $\text{Im}\,\boldsymbol{v}$ are linearly independent. We can get $\boldsymbol{P}$ is invertible, and then get $\boldsymbol{A} = \boldsymbol{P}\boldsymbol{C}\boldsymbol{P}^{-1}$.

To prove that vectors $\text{Re}\,\boldsymbol{v}$ and $\text{Im}\,\boldsymbol{v}$ are linearly independent.

If $\text{Re}\,\boldsymbol{v}$ and $\text{Im}\,\boldsymbol{v}$ are not linearly independent, then $\text{Re}\,\boldsymbol{v}$ must be a multiplicity of $\text{Im}\,\boldsymbol{v}$, say, $\text{Re}\,\boldsymbol{v} = c\,\text{Im}\,\boldsymbol{v}$, $c \neq 0$. Then

$\boldsymbol{A}(\text{Re}\,\boldsymbol{v}) = c\,\boldsymbol{A}(\text{Im}\,\boldsymbol{v})$

$\Rightarrow a\,\text{Re}\,\boldsymbol{v} + b\,\text{Im}\,\boldsymbol{v} = -c\,b\,\text{Re}\,\boldsymbol{v} + c\,a\,\text{Im}\,\boldsymbol{v}$

$\Rightarrow \begin{cases} a = -c\,b \\ b = c\,a \end{cases}$

$\Rightarrow b = c\,a = c\,(-c\,b) = -c^2\,b$

It is impossible for $b \neq 0$.

Thus $\text{Re}\,\boldsymbol{v}$ and $\text{Im}\,\boldsymbol{v}$ are linearly independent.

Exercises of Section 5.5.