4. Vector Spaces

4.1 Vector spaces and subspaces

- Definition Page 33 of Ch.1.

A vector space is a nonempty set $\boldsymbol{V}$ of vectors, on which are defined two operations, called vector addition and multiplication by scalars, subject to the ten axioms (or rules) listed below. The axioms must hold for all vectors $\boldsymbol{u}$, $\boldsymbol{v}$, and $\boldsymbol{w}$ in $\boldsymbol{V}$ and for all scalars c and d.

(1) $\boldsymbol{u} + \boldsymbol{v} \in \boldsymbol{V}$ (closure of addition)
(2) $\boldsymbol{u} + \boldsymbol{v} = \boldsymbol{v} + \boldsymbol{u}$ (interchange of addition)
(3) $(\boldsymbol{u} + \boldsymbol{v}) + \boldsymbol{w} = \boldsymbol{u} + (\boldsymbol{v} + \boldsymbol{w})$ (association of addition)
(4) $\boldsymbol{u} + \boldsymbol{0} = \boldsymbol{u}$ (unit of addition)
(5) $\boldsymbol{u} + (-\boldsymbol{u}) = \boldsymbol{0}$ (inverse of addition)
(6) $c\boldsymbol{u} \in \boldsymbol{V}$ (closure of multiplication)
(7) $c(d\boldsymbol{u}) = (cd)\boldsymbol{u}$ (association of multiplication)
(8) $1\boldsymbol{u} = \boldsymbol{u}$ (unit of multiplication)
(9) $c(\boldsymbol{u} + \boldsymbol{v}) = c\boldsymbol{u} + c\boldsymbol{v}$ (distribution of addition & multiplication)
(10) $(c + d)\boldsymbol{u} = c\boldsymbol{u} + d\boldsymbol{u}$ (distribution of addition & multiplication)

- In other words, a vector space is a set of vectors together with rules for vector addition and scalar multiplication satisfying

  封閉性 (closure for vector addition and scalar multiplication)
  交換律 (interchange)
  結合律 (association)
  單位元素 (unit)
  反元素 (inverse)
  分配律 (distribution)

- Ex.1.

  The spaces $\boldsymbol{R}^n$, where $n \geq 1$, are the premier examples of vector spaces.

subspace

- Definition

  A subspace of a vector space ***V*** is a subset ***H*** of ***V*** such that ***H*** is itself a vector space under the same operations of vector addition and scalar multiplication that are already defined on ***V***.

- By mathematical formula, the subspace is defined as either

  "$\boldsymbol{u}, \boldsymbol{v} \in \boldsymbol{H}$; $a, b \in \boldsymbol{R} \Rightarrow a\,\boldsymbol{u} + b\,\boldsymbol{v} \in \boldsymbol{H}$" $\Rightarrow \boldsymbol{H}$ is a subspace

  or

  (*a*) $\mathbf{0} \in \boldsymbol{H}$ (強調而已，非必要)
  (*b*) $\boldsymbol{u}, \boldsymbol{v} \in \boldsymbol{H} \Rightarrow \boldsymbol{u} + \boldsymbol{v} \in \boldsymbol{H}$
  (*c*) $\boldsymbol{u} \in \boldsymbol{H}$, $c \in \boldsymbol{R} \Rightarrow c\boldsymbol{u} \in \boldsymbol{H}$

  $\Rightarrow$ ***H*** is a subspace

- Ex.6. $\{\mathbf{0}\}$ is a subspace.

- Note:
  (1) Every vector space is a subspace, and every subspace is a vector space.
  (2) The vector space $\boldsymbol{R}^2$ is not a subspace of $\boldsymbol{R}^3$, because $\boldsymbol{R}^2$ is not even a subset of $\boldsymbol{R}^3$.

$$\left\{ \begin{bmatrix} a \\ b \\ 0 \end{bmatrix} : a, b \in R \right\}$$ is not $\boldsymbol{R}^2$ but is a subspace of $\boldsymbol{R}^3$.

- Subspace test
  A subset $\boldsymbol{H}$ of a vector space $\boldsymbol{V}$ is a subspace of $\boldsymbol{V}$ if and only if the following conditions are all satisfied :
  (a) $\mathbf{0}$ (zero vector) $\in \boldsymbol{H}$
  (b) If $\boldsymbol{u}$ and $\boldsymbol{v} \in \boldsymbol{H}$, then $\boldsymbol{u} + \boldsymbol{v} \in \boldsymbol{H}$
  (c) If $\boldsymbol{u} \in \boldsymbol{H}$ and c is any scalar, then $c\boldsymbol{u} \in \boldsymbol{H}$

- Ex.10.
  If $\boldsymbol{v}_1, \boldsymbol{v}_2 \in \boldsymbol{V}$, $\boldsymbol{H} = span\{\boldsymbol{v}_1, \boldsymbol{v}_2\}$, then $\boldsymbol{H}$ is a subspace of $\boldsymbol{V}$.

- Example
  Is $\{(3s, 2 + 5s) \mid s \in \boldsymbol{R}\}$ a subspace?

  Answer. *NO*! The set doesn't contain "**0**".

- A subspace spanned by a set
  It means that a set of vectors spans a subspace.

- Theorem 1
  Given $\boldsymbol{v}_1,\ldots,\boldsymbol{v}_p$ in a vector space $\boldsymbol{V}$,
  the set $span\{\boldsymbol{v}_1,\ldots,\boldsymbol{v}_p\}$ is a subspace of $\boldsymbol{V}$.

  Proof. (Trivial).

- Ex.11.

  Is $\{(a-3b, b-a, a, b) \mid a, b \in \boldsymbol{R}\}$ a subspace ?

  Answer.

  Yes !

  $$(a-3b, b-a, a, b) = \begin{bmatrix} a-3b \\ -a+b \\ a \\ b \end{bmatrix} = a\begin{bmatrix} 1 \\ -1 \\ 1 \\ 0 \end{bmatrix} + b\begin{bmatrix} -3 \\ 1 \\ 0 \\ 1 \end{bmatrix}$$

  The set is spanned by $\begin{bmatrix} 1 \\ -1 \\ 1 \\ 0 \end{bmatrix}$ and $\begin{bmatrix} -3 \\ 1 \\ 0 \\ 1 \end{bmatrix}$.

- Exercises of Section 4.1.

4.2 Null space, column space, & linear transformation

- Definition

  The null space of an $m \times n$ matrix $\boldsymbol{A}$ is the set Nul $\boldsymbol{A}$ of all solutions to the homogeneous equation $\boldsymbol{Ax} = \boldsymbol{0}$.

  In set notation, Nul $\boldsymbol{A} = \{ \boldsymbol{x} \mid \boldsymbol{x} \in \boldsymbol{R}^n$ and $\boldsymbol{Ax} = \boldsymbol{0} \}$.

  Another description of Nul $\boldsymbol{A}$ is the set of all $\boldsymbol{x}$ in $\boldsymbol{R}^n$ that are mapped into the zero vector of $\boldsymbol{R}^m$ via the linear transformation $\boldsymbol{x} \mapsto \boldsymbol{Ax}$.

- Theorem 2

  The null space of an $m \times n$ matrix $\boldsymbol{A}$ is a subspace of $\boldsymbol{R}^n$.

  Equivalently, the set of all solutions to a system $\boldsymbol{A}_{m\times n}\boldsymbol{x} = \boldsymbol{0}$ is a subspace of $\boldsymbol{R}^n$.

  Proof.

  (*i*) $\boldsymbol{0} \in$ Nul $\boldsymbol{A}$

  (*ii*) $\boldsymbol{u}, \boldsymbol{v} \in$ Nul $\boldsymbol{A}$; that is $\boldsymbol{Au} = \boldsymbol{0}$ and $\boldsymbol{Av} = \boldsymbol{0}$, then $\boldsymbol{A}(\boldsymbol{u} + \boldsymbol{v}) = \boldsymbol{Au} + \boldsymbol{Av} = \boldsymbol{0} + \boldsymbol{0} = \boldsymbol{0}$; that is $(\boldsymbol{u} + \boldsymbol{v}) \in$ Nul $\boldsymbol{A}$.

  (*iii*) $\boldsymbol{u} \in$ Nul $\boldsymbol{A}$ and $c \in \boldsymbol{R}$, that is $\boldsymbol{Au} = \boldsymbol{0}$, then $\boldsymbol{A}(c\boldsymbol{u}) = c\boldsymbol{Au} = \boldsymbol{0}$, that is $c\boldsymbol{u} \in$ Nul $\boldsymbol{A}$.

- Ex.2.

  Is $\left\{ \begin{bmatrix} a \\ b \\ c \\ d \end{bmatrix} \middle| \; a-2b+5c=d \text{ and } c-a=b \right\}$ a subspace?

Answer.

Since $\begin{bmatrix} a \\ b \\ c \\ d \end{bmatrix}$ is the general solution of $\begin{bmatrix} 1 & -2 & 5 & -1 \\ -1 & -1 & 1 & 0 \end{bmatrix} \begin{bmatrix} a \\ b \\ c \\ d \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$,

$\left\{ \begin{bmatrix} a \\ b \\ c \\ d \end{bmatrix} \right\}$ is Nul ***A*** and then a subspace .

Ex.2 can also be solved by the same method as Ex.11 of Section 4.1. The solution is $Span\left\{ \begin{bmatrix} -1 \\ 2 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 1 \\ -1 \\ 0 \\ 3 \end{bmatrix} \right\}$

An explicit description of Nul *A*

- Goal: directly find a spanning set for the null space of a matrix.

- Ex.3. Nul ***A*** = ? where ***A*** = $\begin{bmatrix} -3 & 6 & -1 & 1 & -7 \\ 1 & -2 & 2 & 3 & -1 \\ 2 & -4 & 5 & 8 & -4 \end{bmatrix}$

Solution. Nul ***A*** = { ***x*** | ***Ax*** = **0** }

(1) Row reduce the augmented matrix [***A***|**0**] (reduced echelon matrix)

$$\begin{bmatrix} -3 & 6 & -1 & 1 & -7 & 0 \\ 1 & -2 & 2 & 3 & -1 & 0 \\ 2 & -4 & 5 & 8 & -4 & 0 \end{bmatrix} \rightarrow \cdots \rightarrow \begin{bmatrix} 1 & -2 & 0 & -1 & 3 & 0 \\ 0 & 0 & 1 & 2 & -2 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

(2) Transfer the echelon matrix to (solution) equations

$$\begin{cases} x_1 - 2x_2 \qquad - x_4 + 3x_5 = 0 \\ \qquad\qquad x_3 + 2x_4 - 2x_5 = 0 \end{cases}$$

(3) Write the general solution in terms of free variables

$$\begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \\ x_5 \end{bmatrix} = \begin{bmatrix} 2x_2 + x_4 - 3x_5 \\ x_2 \\ -2x_4 + 2x_5 \\ x_4 \\ x_5 \end{bmatrix} = x_2 \begin{bmatrix} 2 \\ 1 \\ 0 \\ 0 \\ 0 \end{bmatrix} + x_4 \begin{bmatrix} 1 \\ 0 \\ -2 \\ 1 \\ 0 \end{bmatrix} + x_5 \begin{bmatrix} -3 \\ 0 \\ 2 \\ 0 \\ 1 \end{bmatrix}$$

(4) $\left\{ \begin{bmatrix} 2 \\ 1 \\ 0 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 1 \\ 0 \\ -2 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} -3 \\ 0 \\ 2 \\ 0 \\ 1 \end{bmatrix} \right\}$ is a spanning set for Nul ***A***.

- Note

  (1) The spanning vectors are linearly independent.

  (2) The number of spanning vectors is the number of free variable.

  (3) The set of the spanning vectors is a basis of Nul ***A***.

The column space of a matrix

- Definition

  The column space of a *m*×*n* matrix ***A*** is the set Col ***A*** of all linear combination of the columns of ***A***.

  In mathematical formula,

  Col $\boldsymbol{A} = \{ \boldsymbol{b} \mid \boldsymbol{b}$ is in $\boldsymbol{R}^m$ and $\boldsymbol{b} = \boldsymbol{A}\boldsymbol{x}$ for some $\boldsymbol{x}$ in $\boldsymbol{R}^n \}$

- Note

  Col ***A*** is the range of the linear transformation $\boldsymbol{x} \mapsto \boldsymbol{A}\boldsymbol{x}$.

- Theorem 3
  The column space of an $m \times n$ matrix $\boldsymbol{A}$ is a subspace of $\boldsymbol{R}^m$.

  proof.

  If $\boldsymbol{A}$ is denoted by $[\ \boldsymbol{a}_1\ \boldsymbol{a}_2\ \dots\ \boldsymbol{a}_n\ ]$,
  then Col $\boldsymbol{A}$ = $span\ \{\ \boldsymbol{a}_1, \boldsymbol{a}_2, \dots, \boldsymbol{a}_n\ \}$.
  By Theorem 1 in Section 4.1 and the number of entries in each vector of Col $\boldsymbol{A}$ is m, thus Col $\boldsymbol{A}$ is a subspace of $\boldsymbol{R}^m$.

- Note
  Col $\boldsymbol{A} = \boldsymbol{R}^m$ for $m \times n$ matrix $\boldsymbol{A}$,

  if and only if

  "the equation $\boldsymbol{Ax} = \boldsymbol{b}$ is consistent for every $\boldsymbol{b}$ in $\boldsymbol{R}^m$ ".

- Ex.7.

  Let $\boldsymbol{A} = \begin{bmatrix} 2 & 4 & -2 & 1 \\ -2 & -5 & 7 & 3 \\ 3 & 7 & -8 & 6 \end{bmatrix}$

  (a) Is $\boldsymbol{u} = \begin{bmatrix} 3 \\ -2 \\ -1 \\ 0 \end{bmatrix} \in$ Nul $\boldsymbol{A}$?

  (b) Is $\boldsymbol{v} = \begin{bmatrix} 3 \\ -1 \\ 3 \end{bmatrix} \in$ Col $\boldsymbol{A}$?

  Answer:

  (a) $\boldsymbol{Au} \neq \boldsymbol{0} \Rightarrow \boldsymbol{u} \notin$ Nul $\boldsymbol{A}$.

  (b) $\boldsymbol{Ax} = \boldsymbol{v} \Rightarrow \boldsymbol{x}$ exists $\Rightarrow \boldsymbol{v} \in$ Col $\boldsymbol{A}$.

- Comparison between Nul $\boldsymbol{A}$ and Col $\boldsymbol{A}$ for an $m \times n$ matrix $\boldsymbol{A}$

Nul $\boldsymbol{A}$	Col $\boldsymbol{A}$
1. Nul $\boldsymbol{A}$ is a subspace of $\boldsymbol{R}^n$	1. Col $\boldsymbol{A}$ is a subspace of R^m
5. $\boldsymbol{Av} = \mathbf{0} \Leftrightarrow \boldsymbol{v} \in$ Nul $\boldsymbol{A}$	5. $\boldsymbol{Ax} = \boldsymbol{v}$ is consistent $\Leftrightarrow$ $\boldsymbol{v} \in$ Col $\boldsymbol{A}$
7. Nul $\boldsymbol{A} = \{\mathbf{0}\} \Leftrightarrow$ $\boldsymbol{Ax} = \mathbf{0}$ has only the trivial solution	7. Col $\boldsymbol{A} = \boldsymbol{R}^m \Leftrightarrow$ $\boldsymbol{Ax} = \boldsymbol{b}$ has a solution for every $\boldsymbol{b}$ in $\boldsymbol{R}^m$
8. Nul $\boldsymbol{A} = \{\mathbf{0}\} \Leftrightarrow$ the linear transformation $\boldsymbol{x} \mapsto \boldsymbol{Ax}$ is one-to-one.	8. Col $\boldsymbol{A} = R^m \Leftrightarrow$ the linear transformation $\boldsymbol{x} \mapsto \boldsymbol{Ax}$ is onto.

- More comparisons are shown in page 222 of textbook.

Kernel and range of a linear transformation

- Definition
  A linear transformation $\boldsymbol{T}$ from a vector space $\boldsymbol{V}$ into a vector space $\boldsymbol{W}$ is a rule that assigns to each vector $\boldsymbol{x}$ in $\boldsymbol{V}$ a unique vector $\boldsymbol{T}(\boldsymbol{x})$ in $\boldsymbol{W}$, such that
  (i) $\boldsymbol{T}(\boldsymbol{u} + \boldsymbol{v}) = \boldsymbol{T}(\boldsymbol{u}) + \boldsymbol{T}(\boldsymbol{v})$ for all $\boldsymbol{u}, \boldsymbol{v}$ in $\boldsymbol{V}$.
  (ii) $\boldsymbol{T}(c\boldsymbol{u}) = c\,\boldsymbol{T}(\boldsymbol{u})$ for all $\boldsymbol{u}$ in $\boldsymbol{V}$ and all scalar c.

- Kernel (or null space) of linear transformation
  $\boldsymbol{T} = \{ \boldsymbol{u} \mid \boldsymbol{T}(\boldsymbol{u}) = \mathbf{0} \}$.

- Range of linear transformation $\boldsymbol{T} = \{ \boldsymbol{T}(\boldsymbol{u}) \mid \boldsymbol{u} \in \boldsymbol{V} \}$.

- If $\boldsymbol{T} : \boldsymbol{x} \mapsto \boldsymbol{Ax}$, then "kernel of $\boldsymbol{T}$ = Nul $\boldsymbol{A}$" and "range of $\boldsymbol{T}$ = Col $\boldsymbol{A}$" .

- Exercises of Section 4.2.

4.3 Linear independent sets; Bases

- Definition

  Let $\boldsymbol{H}$ be a subspace of a vector space $\boldsymbol{V}$. A set of vectors $\beta = \{ \boldsymbol{b}_1, \ldots, \boldsymbol{b}_p\}$ in $\boldsymbol{V}$ is a basis for $\boldsymbol{H}$ if

  (*i*) β is a linearly independent set, and

  (*ii*) the subspace spanned by β is coincides with $\boldsymbol{H}$; that is $\boldsymbol{H} = span\ \{ \boldsymbol{b}_1, \ldots , \boldsymbol{b}_p\}$.

- Ex.3.

  For an invertible matrix $\boldsymbol{A} = [\boldsymbol{a}_1 \ldots \boldsymbol{a}_n]$,

  $\{\boldsymbol{a}_1, \ldots, \boldsymbol{a}_n\}$ is a basis of $\boldsymbol{R}^n$, is also a basis for Col $\boldsymbol{A}$.

- Ex.5.

  Is $\left\{ \begin{bmatrix} 3 \\ 0 \\ -6 \end{bmatrix}, \begin{bmatrix} -4 \\ 1 \\ 7 \end{bmatrix}, \begin{bmatrix} -2 \\ 1 \\ 5 \end{bmatrix} \right\}$ a basis of $\boldsymbol{R}^3$?

  Answer.

  $$\begin{vmatrix} 3 & -4 & -2 \\ 0 & 1 & 1 \\ -6 & 7 & 5 \end{vmatrix} = 6 \neq 0.$$

  It is a basis for $\boldsymbol{R}^3$.

- Note: Minimal set

  A basis is an efficient spanning set that contains no unnecessary vectors.

Bases for Nul *A* and Col *A*

- How to find a basis for Nul ***A*** ?
  (Refer to Section 4.2 on page 10).
  How to find a basis for Col ***A*** ?

- Ex.8.
  Find a basis for Col ***B***, where $\boldsymbol{B} = \begin{bmatrix} 1 & 4 & 0 & 2 & 0 \\ 0 & 0 & 1 & -1 & 0 \\ 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}$

  Solution:
  Every nonpivot column of ***B*** is linear combination of the pivot columns. That is nonpivot columns are redundant for Col ***B***. Thus, a basis for

  $$\text{Col } \boldsymbol{B} = \left\{ \begin{bmatrix} 1 \\ 0 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 0 \\ 1 \\ 0 \end{bmatrix} \right\} \qquad .$$

- Note
  Elementary row operations on a matrix do not affect the linear dependence relations among the columns of the matrix

- Ex. 9. $\boldsymbol{A} = \begin{bmatrix} 1 & 4 & 0 & 2 & -1 \\ 3 & 12 & 1 & 5 & 5 \\ 2 & 8 & 1 & 3 & 2 \\ 5 & 20 & 0 & 8 & 8 \end{bmatrix} \xleftrightarrow{\text{row equivalent}} \begin{bmatrix} 1 & 4 & 0 & 2 & 0 \\ 0 & 0 & 1 & -1 & 0 \\ 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix} = \boldsymbol{B}$

  $$\text{Thus } \left\{ \begin{bmatrix} 1 \\ 3 \\ 2 \\ 5 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} -1 \\ 5 \\ 2 \\ 8 \end{bmatrix} \right\} \text{ is a basis for Col } \boldsymbol{A}.$$

- Conclusion (Theorem 6)
  The pivot columns of a matrix ***A*** form a basis for Col ***A*** .

- Very Important

  Be careful to use pivot columns of ***A*** itself for the basis of Col ***A***. The columns of an echelon form ***B*** are often not in the Col ***A*** .

  For instance, in Ex.9.

$$\left\{ \begin{bmatrix} 1 \\ 3 \\ 2 \\ 5 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} -1 \\ 5 \\ 2 \\ 8 \end{bmatrix} \right\} \text{ is a basis of Col } \boldsymbol{A} \text{, but}$$

$$\left\{ \begin{bmatrix} 1 \\ 0 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 0 \\ 1 \\ 0 \end{bmatrix} \right\} \text{ is not a basis of Col } \boldsymbol{A}.$$

- Note:

  (*i*) Basis of subspace is not unique.

  (*ii*) A basis is a maximal independent set.

  (*iii*) A basis is a minimal spanning set.

  For example,

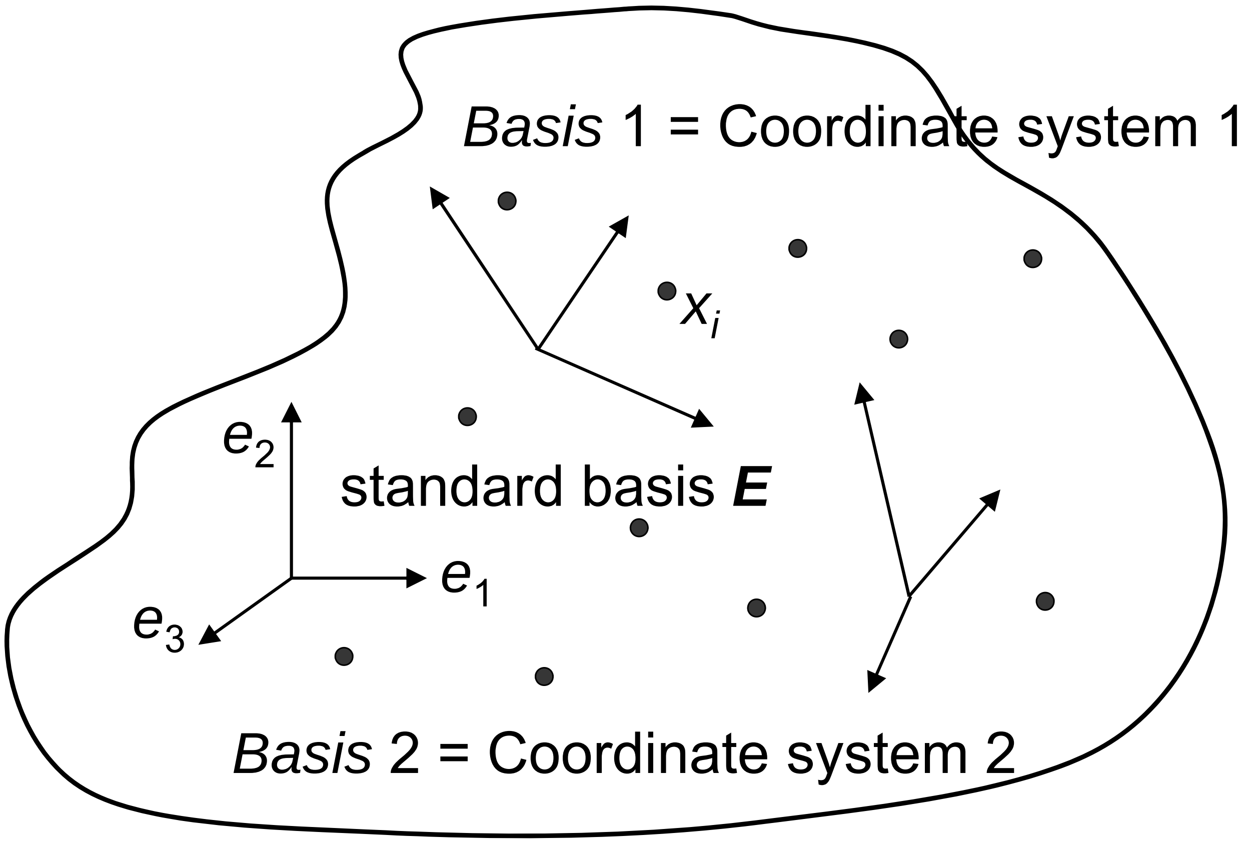
$$\left\{ \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} \right\} \text{ and } \left\{ \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix} \right\} \text{ are both bases of } \boldsymbol{R}^3.$$

- Exercises of Section 4.3.

4.4 Coordinate systems

- Goal
  Using coordinate system to specify a compact basis β for a vector space ***V***; for example,
  1. A vector space spanned by $\left\{\begin{bmatrix}1\\2\\0\end{bmatrix},\begin{bmatrix}1\\0\\0\end{bmatrix}\right\}$ can be described by $\left\{\begin{bmatrix}1\\0\end{bmatrix},\begin{bmatrix}0\\1\end{bmatrix}\right\}$.
  2. The basis $\left\{\begin{bmatrix}0\\1\end{bmatrix},\begin{bmatrix}1\\0\end{bmatrix}\right\}$ can be replaced by $\left\{\begin{bmatrix}1\\2\end{bmatrix},\begin{bmatrix}3\\4\end{bmatrix}\right\}$.
- Theorem 7 (The unique representation theorem)
  Let $\beta = \{\boldsymbol{b}_1, \ldots, \boldsymbol{b}_n\}$ be a basis for a vector space ***V***. Then for each ***x*** in ***V***, there exists a unique set of scalars $c_1, \ldots, c_n$ such that $\boldsymbol{x} = c_1\boldsymbol{b}_1 + \ldots + c_n\boldsymbol{b}_n$.

H is a subspace of $\boldsymbol{R}^n$ (take n = 3 as one example)



- Definition

  Suppose the set $\beta = \{\boldsymbol{b}_1, \ldots, \boldsymbol{b}_n\}$ is a basis for $\boldsymbol{V}$ and $\boldsymbol{x}$ is in $\boldsymbol{V}$. The coordinates of $\boldsymbol{x}$ relative to the basis β (or β-coordinates of $\boldsymbol{x}$) are the weights $c_1, \ldots, c_n$ such that $\boldsymbol{x} = c_1\boldsymbol{b}_1 + \ldots + c_n\boldsymbol{b}_n$.

- If $c_1, \ldots, c_n$ are the β-coordinates of $\boldsymbol{x}$, then the vector in $\boldsymbol{R}^n$

  $$[x]_{\boldsymbol{\beta}} = \begin{bmatrix} c_1 \\ \vdots \\ c_n \end{bmatrix}$$ is the coordinate vector of $\boldsymbol{x}$ (relative to β, or the β-coordinate vector of $\boldsymbol{x}$.

- In general cases, the entries in a vector are the coordinates relative to the standard basis $\boldsymbol{E}$. For example,

  $$\begin{bmatrix} 2 \\ 5 \end{bmatrix} = 2\begin{bmatrix} 1 \\ 0 \end{bmatrix} + 5\begin{bmatrix} 0 \\ 1 \end{bmatrix} = 2\mathbf{e}_1 + 5\mathbf{e}_2 \Rightarrow \boldsymbol{E} = \{\mathbf{e}_1, \mathbf{e}_2\}.$$

- Ex.1.

  Consider a basis $\beta = \{\boldsymbol{b}_1, \boldsymbol{b}_2\}$ for $\boldsymbol{R}^2$, where

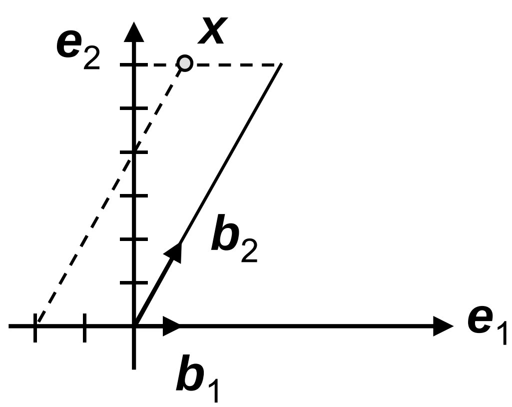
  $\boldsymbol{b}_1 = \begin{bmatrix} 1 \\ 0 \end{bmatrix}$ and $\boldsymbol{b}_2 = \begin{bmatrix} 1 \\ 2 \end{bmatrix}$.

  Suppose an $\boldsymbol{x}$ in $\boldsymbol{R}^2$ has the coordinate vector $[\boldsymbol{x}]_\beta = \begin{bmatrix} -2 \\ 3 \end{bmatrix}$. Find $\boldsymbol{x}$.
  (that is to find $[\boldsymbol{x}]_E$)

  Answer.

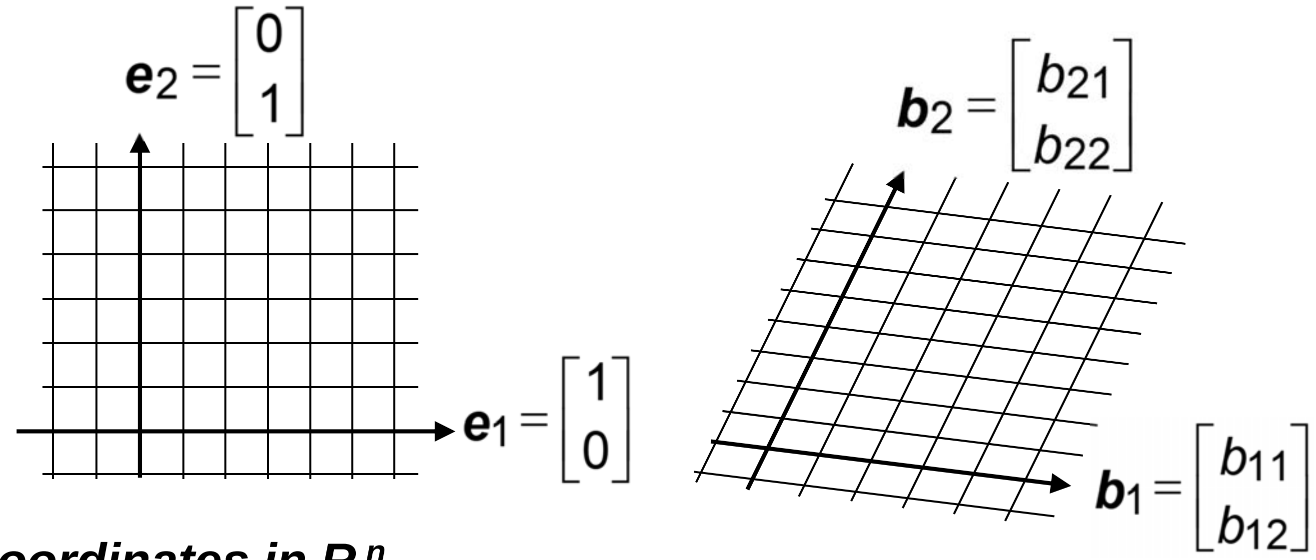
  $$\boldsymbol{x} = -2\boldsymbol{b}_1 + 3\boldsymbol{b}_2 = -2\begin{bmatrix} 1 \\ 0 \end{bmatrix} + 3\begin{bmatrix} 1 \\ 2 \end{bmatrix} = \begin{bmatrix} 1 \\ 6 \end{bmatrix}$$

  That is, $[\boldsymbol{x}]_{\boldsymbol{\beta}} \rightarrow [\boldsymbol{x}]_{\boldsymbol{E}}$.



A graphic interpretation of coordinates

- A coordinate system on a set consists of a one-to-one mapping of the points in the set into $\boldsymbol{R}^n$; for example,



Coordinates in $\boldsymbol{R}^n$

- When a basis β for $\boldsymbol{R}^n$ is fixed, the β–coordinate vector of a specified $\boldsymbol{x}$ is easily found, as in the next example.

- Ex.4.

$$\text{Let } \boldsymbol{b}_1 = \begin{bmatrix} 2 \\ 1 \end{bmatrix},\ \boldsymbol{b}_2 = \begin{bmatrix} -1 \\ 1 \end{bmatrix},\ \boldsymbol{x} = \begin{bmatrix} 4 \\ 5 \end{bmatrix},\ \text{and } \boldsymbol{\beta} = \{\boldsymbol{b}_1, \boldsymbol{b}_2\}.$$

Find the coordinate vector $[\boldsymbol{x}]_\beta$ of relative to β
(*i.e.,* $[\boldsymbol{x}]_E \to [\boldsymbol{x}]_\beta$).

Solution.

$$[\boldsymbol{x}]_{\boldsymbol{E}} = c_1\boldsymbol{b}_1 + c_2\boldsymbol{b}_2$$

$$\Rightarrow \begin{bmatrix} 4 \\ 5 \end{bmatrix}_{\boldsymbol{E}} = c_1\begin{bmatrix} 2 \\ 1 \end{bmatrix} + c_2\begin{bmatrix} -1 \\ 1 \end{bmatrix} = \begin{bmatrix} 2 & -1 \\ 1 & 1 \end{bmatrix}\begin{bmatrix} c_1 \\ c_2 \end{bmatrix}_{\boldsymbol{\beta}}$$

$$\Rightarrow \begin{bmatrix} c_1 \\ c_2 \end{bmatrix}_{\boldsymbol{\beta}} = \begin{bmatrix} \frac{1}{3} & \frac{1}{3} \\ \frac{-1}{3} & \frac{2}{3} \end{bmatrix}\begin{bmatrix} 4 \\ 5 \end{bmatrix}_{\boldsymbol{E}} = \begin{bmatrix} 3 \\ 2 \end{bmatrix}_{\boldsymbol{\beta}}$$

- The matrix for changing the β-coordinates of a vector $\boldsymbol{x}$ into the standard coordinates for $\boldsymbol{x}$ is called the change-of-coordinates matrix $\boldsymbol{P}_\beta$. If the basis $\beta = \{\boldsymbol{b}_1, \dots, \boldsymbol{b}_n\}$, then $\boldsymbol{P}_\beta = [\,\boldsymbol{b}_1, \dots, \boldsymbol{b}_n]$. That is, $[\boldsymbol{x}]_E = \boldsymbol{P}_\beta\, [\boldsymbol{x}]_\beta$.

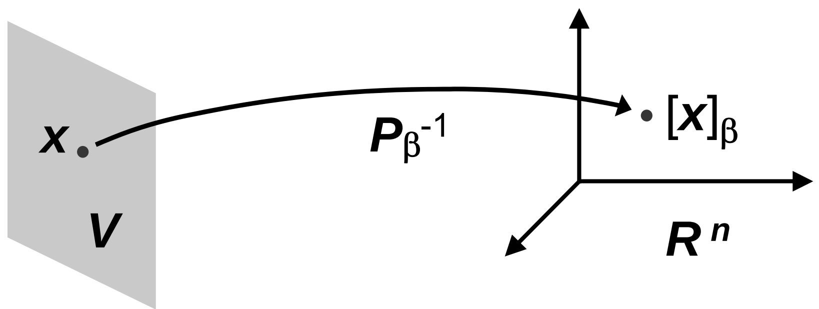
  Since the columns of $\boldsymbol{P}_\beta$ form a basis for $\boldsymbol{R}^n$, $\boldsymbol{P}_\beta$ is invertible.

  Thus $\boldsymbol{P}_\beta^{-1}\, [\boldsymbol{x}]_E = [\boldsymbol{x}]_\beta$

  For example, Ex.1 and Ex.4.

The coordinate mapping

- Choosing a basis $\beta = \{\boldsymbol{b}_1, \dots, \boldsymbol{b}_n\}$ for a vector space $\boldsymbol{V}$. The coordinate mapping $\boldsymbol{x} \mapsto [\boldsymbol{x}]_\beta$ is a one-to-one linear transformation from $\boldsymbol{V}$ onto $\boldsymbol{R}^n$.



- Theorem 8

  Let $\beta = \{\boldsymbol{b}_1, \dots, \boldsymbol{b}_n\}$ be a basis for a vector space $\boldsymbol{V}$. Then the coordinate mapping $\boldsymbol{x} \mapsto [\boldsymbol{x}]_\beta$ is a one-to-one linear transformation from $\boldsymbol{V}$ onto $\boldsymbol{R}^n$.

- The coordinate mapping in Theorem 8 is an important example of an isomorphism from $\boldsymbol{V}$ onto $\boldsymbol{R}^n$. In general, a one-to-one linear transformation from a vector space $\boldsymbol{V}$ onto a vector space $\boldsymbol{W}$ is called an isomorphism from $\boldsymbol{V}$ onto $\boldsymbol{W}$.

- Ex.7. Let $\mathbf{v}_1 = \begin{bmatrix} 3 \\ 6 \\ 2 \end{bmatrix}$, $\mathbf{v}_2 = \begin{bmatrix} -1 \\ 0 \\ 1 \end{bmatrix}$, $\mathbf{x} = \begin{bmatrix} 3 \\ 12 \\ 7 \end{bmatrix}$, and $\boldsymbol{\beta} = \{\mathbf{v}_1, \mathbf{v}_2\}$.

  Determine if $\mathbf{x}$ is in Span$\{\mathbf{v}_1, \mathbf{v}_2\}$, and if it is, find the coordinate vector of $\mathbf{x}$ relative to $\boldsymbol{\beta}$.

  Solution.

  If $\mathbf{x}$ is in $Span\{\mathbf{v}_1, \mathbf{v}_2\}$, then the equation $c_1 \begin{bmatrix} 3 \\ 6 \\ 2 \end{bmatrix} + c_2 \begin{bmatrix} -1 \\ 0 \\ 1 \end{bmatrix} = \begin{bmatrix} 3 \\ 12 \\ 7 \end{bmatrix}$

  is consistent. c_1 and c_2 can be found as 2 and 3,

  thus $\mathbf{x}$ is in $Span\{\mathbf{v}_1, \mathbf{v}_2\}$ and $[\mathbf{x}]_{\boldsymbol{\beta}} = \begin{bmatrix} 2 \\ 3 \end{bmatrix}$.

- Exercises of Section 4.4.

4.5 The dimension of a vector space

- Theorem 9 (The concept of dimension)
  If a vector space ***V*** has a basis β = {b_1, …, b_n} , then any set in ***V*** containing more than *n* vectors must be linearly dependent.

- Theorem 10
  If a vector space ***V*** has a basis of *n* vectors, then every basis of ***V*** must consist of exactly *n* vectors.

- Definition
  If vector space ***V*** is spanned by a finite set, then ***V*** is said to be finite-dimensional, and dimension of ***V***, written as dim ***V***, is the number of vectors in a basis for ***V***. The dimension of the zero vector space { **0** } is defined to be zero. If ***V*** is not spanned by a finite set, then ***V*** is said to be infinite-dimensional.

- EX.1.

  The standard basis for $\boldsymbol{R}^n$ contains n vectors, so dim $\boldsymbol{R}^n = n$.

- Ex.2.

  Let $\boldsymbol{H}$ = *Span* $\{\boldsymbol{v}_1, \boldsymbol{v}_2\}$, where $\boldsymbol{v}_1 = \begin{bmatrix} 3 \\ 6 \\ 2 \end{bmatrix}$ and $\boldsymbol{v}_2 = \begin{bmatrix} -1 \\ 0 \\ 1 \end{bmatrix}$.

  Since $\boldsymbol{v}_1$ and $\boldsymbol{v}_2$ are linearly independent, dim $\boldsymbol{H}$ = 2.

- Ex.3. Find the dimension of the subspace

$$\left\{ \begin{bmatrix} a-3b+6c \\ 5a+4d \\ b-2c-d \\ 5d \end{bmatrix} \middle|\ a, b, c, d \in \boldsymbol{R} \right\}.$$

Solution.

$$\begin{bmatrix} a-3b+6c \\ 5a+4d \\ b-2c-d \\ 5d \end{bmatrix} = a\begin{bmatrix} 1 \\ 5 \\ 0 \\ 0 \end{bmatrix} + b\begin{bmatrix} -3 \\ 0 \\ 1 \\ 0 \end{bmatrix} + c\begin{bmatrix} 6 \\ 0 \\ -2 \\ 0 \end{bmatrix} + d\begin{bmatrix} 0 \\ 4 \\ -1 \\ 5 \end{bmatrix}$$

$\qquad\qquad\qquad\qquad \boldsymbol{v}_1 \qquad \boldsymbol{v}_2 \qquad \boldsymbol{v}_3 \qquad \boldsymbol{v}_4$

$$[\boldsymbol{v}_1 \boldsymbol{v}_2 \boldsymbol{v}_3 \boldsymbol{v}_4] \xrightarrow{\text{row reduction}} \begin{bmatrix} 1 & -3 & 6 & 0 \\ 0 & 1 & -2 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

Thus dim $\boldsymbol{H} = 3$.

Subspaces of a finite-dimensional space

- Theorem 11
  Let ***H*** be a subspace of a finite-dimensional vector space ***V***. Any linearly independent set in ***H*** can be expanded to a basis for ***H***. Also, ***H*** is finite-dimensional and dim ***H*** ≤ dim ***V***.
- Theorem 12
  Let ***V*** be a p-dimensional vector space, $p \geq 1$. Any linearly independent set of exactly p elements in ***V*** is automatically a basis for ***V***. Any set of exactly p elements that spans ***V*** is automatically a basis for ***V*** .
- Theorem 13
  The dimension of Nul ***A*** is the number of free variables in the equation $\boldsymbol{Ax} = \mathbf{0}$, and the dimension of Col ***A*** is the number of pivot columns in ***A*** .
- Exercises for Section 4.5.

4.6 Rank

- Definition

  The row space of an $m \times n$ matrix ***A*** is the set Row ***A*** of all linear combinations of the row vectors of ***A*** .

  $$\text{Row } \boldsymbol{A} = \{ \boldsymbol{b} \mid \boldsymbol{b} \in \boldsymbol{R}^n \text{ and } \boldsymbol{b} = c_1\boldsymbol{r}_1 + \ldots + c_m\boldsymbol{r}_m, \; c_i \in \boldsymbol{R} \text{ and } \boldsymbol{r_i} \text{ is the } i \text{ th row of } \boldsymbol{A} \}$$

  or $\text{Row } \boldsymbol{A} = Span\, \{\boldsymbol{r}_1 \ldots \boldsymbol{r}_m\}$.

- Note
  Row space of $\boldsymbol{A}_{m\times n}$ is a subspace of $\boldsymbol{R}^n$.

$$m\left\{\overbrace{\boldsymbol{A}}^{n}\right.$$

- Theorem 13
  A and ***B*** are row equivalent, then their row spaces are the same. If ***B*** is the echelon matrix of ***A***, then the nonzero rows of ***B*** form a basis for Row ***B*** as well as Row ***A*** .

  Proof.
  A $\rightarrow$ row reduce $\rightarrow$ ***B***
  $\Rightarrow$ the rows of ***B*** are linear combinations of the rows of ***A***
  $\Rightarrow$ the linear combination of rows of ***B*** = a linear combination of rows of ***A***.
  $\Rightarrow$ Row ***B*** $\subseteq$ Row ***A***.
  Since row operations are reversible, ***B*** $\rightarrow$ row reduce $\rightarrow$ ***A***, by the same way, Row ***A*** $\subseteq$ Row ***B***.
  Therefore Row ***A*** = Row ***B***.

- If ***B*** is in echelon form, its nonzero rows are linearly independent. Thus the nonzero rows of ***B*** form a basis of Row ***B*** and Row ***A***.

- Note
  If ***B*** is an echelon matrix from ***A***, then Row ***A*** = Row ***B***, but Col ***A*** $\neq$ Col ***B*** .

- Ex.2. Find bases for Row ***A***, Col ***A***, and Nul ***A*** for

$$\boldsymbol{A} = \begin{bmatrix} -2 & -5 & 8 & 0 & -17 \\ 1 & 3 & -5 & 1 & 5 \\ 3 & 11 & -19 & 7 & 1 \\ 1 & 7 & -13 & 5 & -3 \end{bmatrix}.$$

Solution.

$$\boldsymbol{A} \rightarrow \text{row reduce} \rightarrow \boldsymbol{B} = \begin{bmatrix} 1 & 3 & -5 & 1 & 5 \\ 0 & 1 & -2 & 2 & -7 \\ 0 & 0 & 0 & -4 & 20 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

↑ ↑ ↑ (columns 1, 2, 4)
pivot columns

Row ***A*** = { (1, 3, -5, 1, 5), (0, 1, -2, 2, -7), (0, 0, 0, -4, 20)}

$$\text{Col}\,\boldsymbol{A} = \left\{ \begin{bmatrix} -2 \\ 1 \\ 3 \\ 1 \end{bmatrix} \begin{bmatrix} -5 \\ 3 \\ 11 \\ 7 \end{bmatrix} \begin{bmatrix} 0 \\ 1 \\ 7 \\ 5 \end{bmatrix} \right\}.$$

$$\boldsymbol{B} \xrightarrow{\text{row reduce}} \boldsymbol{C} = \begin{bmatrix} 1 & 0 & 1 & 0 & 1 \\ 0 & 1 & -2 & 0 & 3 \\ 0 & 0 & 0 & 1 & -5 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix} = \begin{cases} x_1 + x_3 + x_5 = 0 \\ x_2 - 2x_3 + 3x_5 = 0 \\ x_4 - 5x_5 = 0 \end{cases}$$

$$\Rightarrow \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \\ x_5 \end{bmatrix} = \begin{bmatrix} -x_3 - x_5 \\ 2x_3 - 3x_5 \\ x_3 \\ 5x_5 \\ x_5 \end{bmatrix} = x_3 \begin{bmatrix} -1 \\ 2 \\ 1 \\ 0 \\ 0 \end{bmatrix} + x_5 \begin{bmatrix} -1 \\ -3 \\ 0 \\ 5 \\ 1 \end{bmatrix} \Rightarrow \text{Nul}\,\boldsymbol{A} = \left\{ \begin{bmatrix} -1 \\ 2 \\ 1 \\ 0 \\ 0 \end{bmatrix} \begin{bmatrix} -1 \\ -3 \\ 0 \\ 5 \\ 1 \end{bmatrix} \right\}.$$

The rank theorem

- Definition
  The rank of ***A*** is the dimension of the column space of ***A*** .

- Note
  Row ***A*** = Col $\boldsymbol{A}^T$.
  The dimension of Row ***A***
  = The dimension of Col $\boldsymbol{A}^T$
  = The rank of $\boldsymbol{A}^T$.

- Definition
  The nullity of ***A*** is the dimension of Nul ***A*** .

Rank = dim Col ***A***
Nullity = dim Nul ***A***

- Theorem 14 (Rank theorem)
  A is an $m\times n$ matrix.
  (*i*) dim (Row ***A***) = dim (Col ***A***) = rank (***A***)
  (*ii*) rank (***A***) + dim (Nul ***A***) = *n* .

  Proof.
  i) rank (***A***) = dim (Col ***A***) = number of pivot columns in ***A***.
  If ***B*** is an echelon matrix of ***A***, then
  dim (Row ***A***) = dim (Row ***B***) = number of nonzero row in ***B***
  = number of pivots in ***A***.
  Thus, dim (Row ***A***) = rank (***A***) = dim (Col ***A***) .

  ii) dim (Nul ***A***) = number of free variables in $\boldsymbol{Ax} = \mathbf{0}$.
  rank (***A***) = dim (Col ***A***) = number of basic variable in $\boldsymbol{Ax} = \mathbf{0}$.
  Since "number of free variables" + "number of basic variables" = *n*,
  rank (***A***) + dim (Nul ***A***) = *n*.

- Note
  Row ***A*** and Nul ***A*** have only the zero vector in common and are actually "perpendicular" to each other.

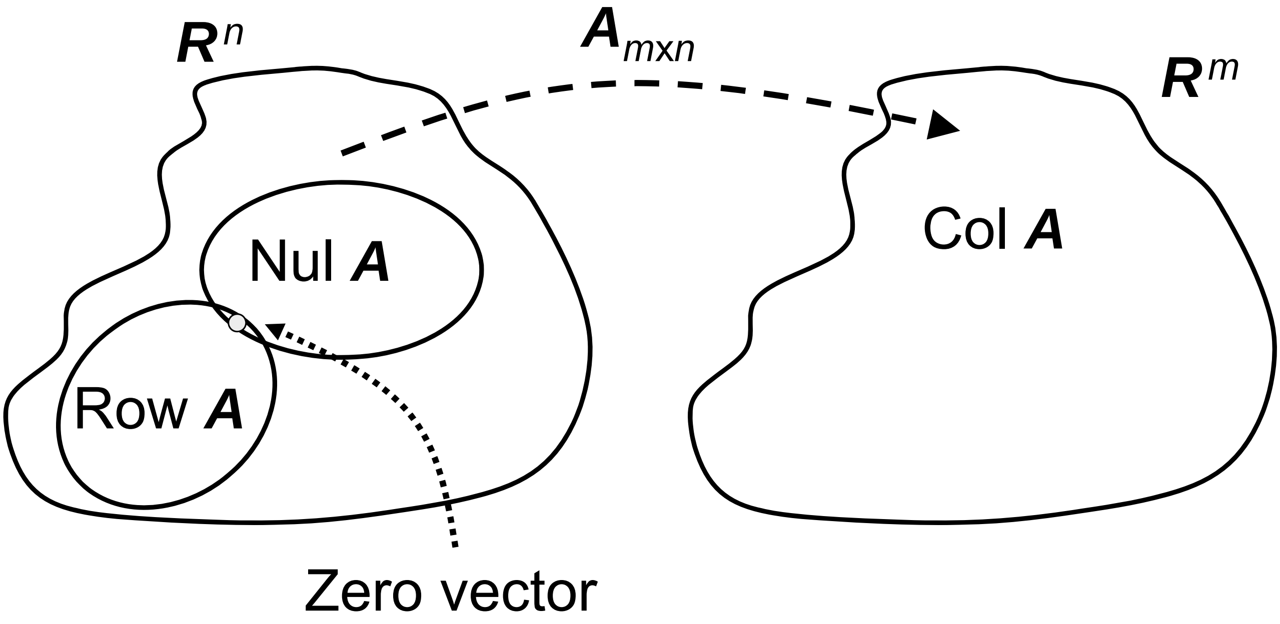
- Ex.4.

$$\boldsymbol{A} = \begin{bmatrix} 3 & 0 & -1 \\ 3 & 0 & -1 \\ 4 & 0 & 5 \end{bmatrix} \qquad \boldsymbol{Ax} = \boldsymbol{0} \equiv \begin{bmatrix} \cdot\cdot & \boldsymbol{r}_1 & \cdot\cdot \\ \cdot\cdot & \boldsymbol{r}_2 & \cdot\cdot \\ \cdot\cdot & \boldsymbol{r}_3 & \cdot\cdot \end{bmatrix} \begin{bmatrix} \vdots \\ \boldsymbol{x}_i \\ \vdots \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

$$\boldsymbol{A} \xrightarrow{\text{row reduce}} \boldsymbol{B} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{bmatrix} \Rightarrow \begin{cases} x_1 = 0 \\ x_3 = 0 \end{cases} \Rightarrow \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = x_2 \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}$$

$$\text{Row } \boldsymbol{A} = \{(1,0,0),(0,0,1)\}$$

$$\text{Nul } \boldsymbol{A} = \left\{ \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix} \right\}.$$



Row $A \perp$ Nul A

Rank and the invertible matrix theorem

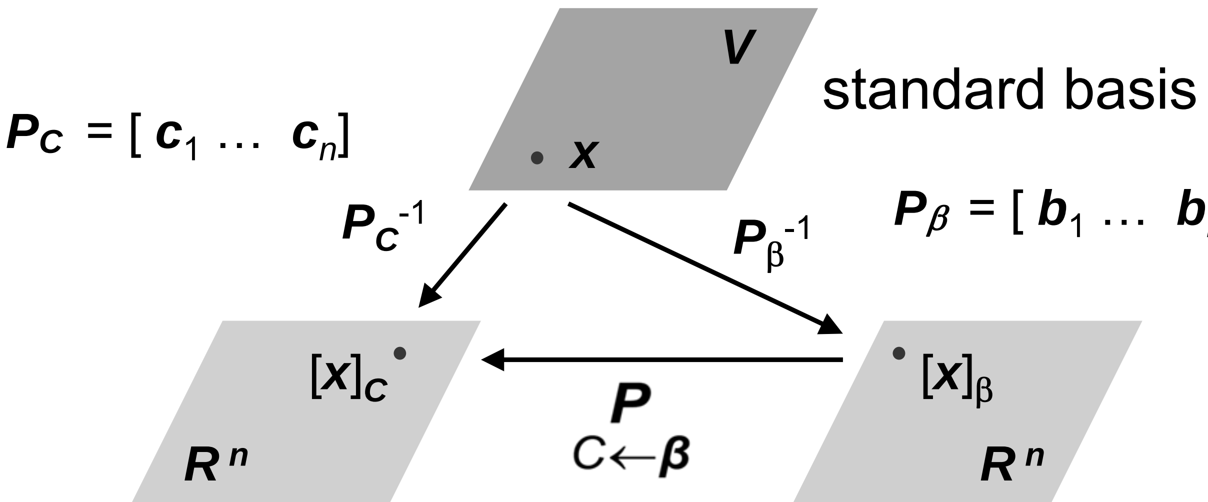
- Theorem (The invertible matrix theorem continued)
  A is an $n \times n$ matrix. The following statements are each equivalent
  (*a*) ***A*** is invertible.
  (*m*) The columns of ***A*** form a basis of R^n.
  (*n*) Col ***A*** = $\boldsymbol{R}^n$.
  (*o*) dim (Col ***A***) = *n*.
  (*p*) rank ***A*** = *n*.
  (*q*) Nul ***A*** = {**0**}.
  (*r*) dim (Nul ***A***) = 0.

  Page 17 in Section 2.3

- Exercises of Section 4.6.

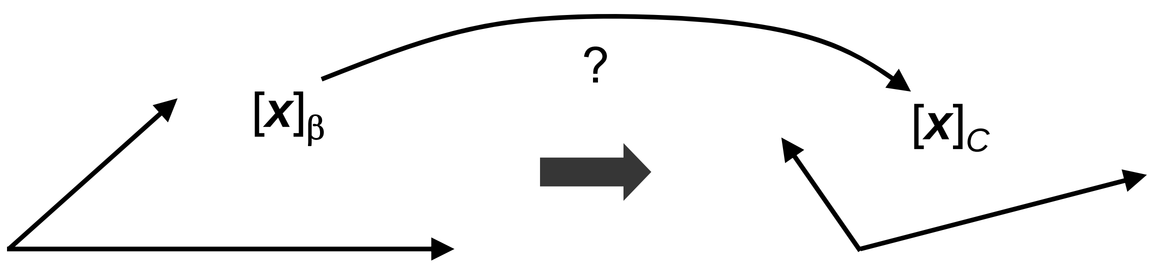
4.7 Change of basis

- Two kinds of changing basis are discussed:
  1) changing between standard basis and non-standard basis. (which has been discussed in Section 4.4.)
  2) changing between two non-standard bases. (will been discussed in this section.)



Changing between two non-standard bases

- Purpose
  In some applications, a problem is described initially using a basis β, but the problem's solution is aided by changing β to a new basis *C*. Here we want to study how to change $[\boldsymbol{x}]_\beta$ to the corresponding $[\boldsymbol{x}]_C$ for each $\boldsymbol{x}$ in $\boldsymbol{V}$.
  For example,



From the known relationship of coordinates β and *C* to find the new coordinates $[\boldsymbol{x}]_C$.

- Ex.1.

  Consider two bases $\beta = \{\boldsymbol{b}_1, \boldsymbol{b}_2\}$ and $\boldsymbol{C} = \{\boldsymbol{c}_1, \boldsymbol{c}_2\}$ for a vector space $\boldsymbol{V}$, such that $\boldsymbol{b}_1 = 4\,\boldsymbol{c}_1 + \boldsymbol{c}_2$ and $\boldsymbol{b}_2 = -6\,\boldsymbol{c}_1 + \boldsymbol{c}_2$.

  Suppose that $\boldsymbol{x} = 3\boldsymbol{b}_1 + \boldsymbol{b}_2$; that is, suppose that $[\boldsymbol{x}]_\beta = \begin{bmatrix} 3 \\ 1 \end{bmatrix}$.
  Find $[\boldsymbol{x}]_C$.

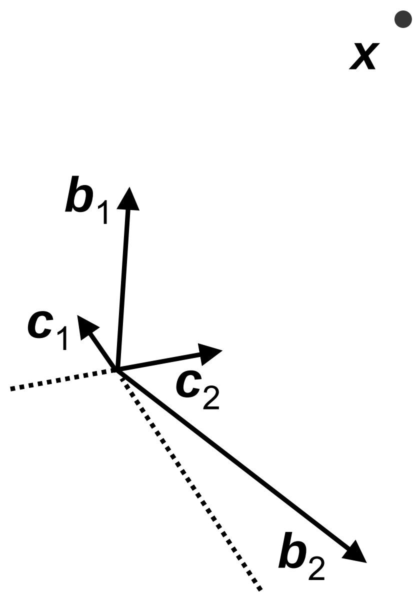
  Solution.

  $$[\boldsymbol{x}]_C = [\, 3\boldsymbol{b}_1 + \boldsymbol{b}_2 \,]_C = 3[\boldsymbol{b}_1]_C + [\boldsymbol{b}_2]_C$$

  $$\Rightarrow [\boldsymbol{x}]_C = [\, [\boldsymbol{b}_1]_C \;\; [\boldsymbol{b}_2]_C \,] \begin{bmatrix} 3 \\ 1 \end{bmatrix}$$

  Since $[\boldsymbol{b}_1]_C = \begin{bmatrix} 4 \\ 1 \end{bmatrix}$ and $[\boldsymbol{b}_2]_C = \begin{bmatrix} -6 \\ 1 \end{bmatrix}$

  $$\Rightarrow [\boldsymbol{x}]_C = \begin{bmatrix} 4 & -6 \\ 1 & 1 \end{bmatrix} \begin{bmatrix} 3 \\ 1 \end{bmatrix}_{\boldsymbol{\beta}} = \begin{bmatrix} 6 \\ 4 \end{bmatrix}_C .$$



- Theorem 15

  Let $\beta = \{\boldsymbol{b}_1, \ldots, \boldsymbol{b}_n\}$ and $\boldsymbol{C} = \{\boldsymbol{c}_1, \ldots, \boldsymbol{c}_n\}$ be bases of a vector space $\boldsymbol{V}$.

  Then there is an $n \times n$ matrix $\underset{C \leftarrow \boldsymbol{\beta}}{P}$ such that $[\boldsymbol{x}]_C = \underset{C \leftarrow \boldsymbol{\beta}}{P} [\boldsymbol{x}]_\beta$.

  The columns of $\underset{C \leftarrow \boldsymbol{\beta}}{P}$ are the $\boldsymbol{C}$-coordinate vectors of the vectors in the basis β. That is , $\underset{C \leftarrow \boldsymbol{\beta}}{P} = [\, [\boldsymbol{b}_1]_C \ldots [\boldsymbol{b}_n]_C \,]$.

- The matrix $\underset{C \leftarrow \boldsymbol{\beta}}{P}$ in Theorem 15 is called the change-of-coordinates matrix from β to $\boldsymbol{C}$. Multiplication by $\underset{C \leftarrow \boldsymbol{\beta}}{P}$ converts β coordinates into C coordinates. That is , $\underset{C \leftarrow \boldsymbol{\beta}}{P} [\boldsymbol{x}]_\beta = [\boldsymbol{x}]_C$.

- The columns of $\underset{C \leftarrow \boldsymbol{\beta}}{P}$ are linearly independent because they are the coordinate vectors of linearly independent set β. (That is, coordinate mapping is one-to-one.)

  Thus, $(\underset{C \leftarrow \boldsymbol{\beta}}{P})^{-1} [\boldsymbol{x}]_C = [\boldsymbol{x}]_\beta$; that is $(\underset{C \leftarrow \boldsymbol{\beta}}{P})^{-1} = \underset{\boldsymbol{\beta} \leftarrow C}{P}$.

Ex.2.

Let $\boldsymbol{b}_1=\begin{bmatrix}-9\\1\end{bmatrix}$, $\boldsymbol{b}_2=\begin{bmatrix}-5\\-1\end{bmatrix}$, $\boldsymbol{c}_1=\begin{bmatrix}1\\-4\end{bmatrix}$, $\boldsymbol{c}_2=\begin{bmatrix}3\\-5\end{bmatrix}$, and

consider the bases for $\boldsymbol{R}^2$ given by $\beta = \{\boldsymbol{b}_1, \boldsymbol{b}_2\}$ and $\boldsymbol{C} = \{c_1, c_2\}$. Find the change-of-coordinates matrix from β to $\boldsymbol{C}$.

Solution.

$$\underset{C\leftarrow\boldsymbol{\beta}}{\boldsymbol{P}}=\underset{C\leftarrow E}{\boldsymbol{P}}\ \underset{E\leftarrow\boldsymbol{\beta}}{\boldsymbol{P}}=\left(\underset{E\leftarrow C}{\boldsymbol{P}}\right)^{-1}\underset{E\leftarrow\boldsymbol{\beta}}{\boldsymbol{P}}=\begin{bmatrix}1&3\\-4&-5\end{bmatrix}^{-1}\begin{bmatrix}-9&-5\\1&-1\end{bmatrix}=\frac{1}{7}\begin{bmatrix}-5&-3\\4&1\end{bmatrix}\begin{bmatrix}-9&-5\\1&-1\end{bmatrix}=\begin{bmatrix}6&4\\-5&-3\end{bmatrix}.$$

Another method

$$\left[\underset{E\leftarrow C}{\boldsymbol{P}}\ \vdots\ \underset{E\leftarrow\boldsymbol{\beta}}{\boldsymbol{P}}\right]\rightarrow\left(\underset{E\leftarrow C}{\boldsymbol{P}}\right)^{-1}\left[\underset{E\leftarrow C}{\boldsymbol{P}}\ \vdots\ \underset{E\leftarrow\boldsymbol{\beta}}{\boldsymbol{P}}\right]\rightarrow\left[\boldsymbol{I}\ \vdots\ \left(\underset{E\leftarrow C}{\boldsymbol{P}}\right)^{-1}\underset{E\leftarrow\boldsymbol{\beta}}{\boldsymbol{P}}\right]$$

$$\begin{bmatrix}1&3&:&-9&-5\\-4&-5&:&1&-1\end{bmatrix}\rightarrow\begin{bmatrix}1&3&:&-9&-5\\0&7&:&-35&-21\end{bmatrix}\rightarrow\begin{bmatrix}1&3&:&-9&-5\\0&1&:&-5&-3\end{bmatrix}\rightarrow\begin{bmatrix}1&0&:&6&4\\0&1&:&-5&-3\end{bmatrix}.$$

Ex.3.

Let $\boldsymbol{b}_1=\begin{bmatrix}1\\-3\end{bmatrix}$, $\boldsymbol{b}_2=\begin{bmatrix}-2\\4\end{bmatrix}$, $\boldsymbol{c}_1=\begin{bmatrix}-7\\9\end{bmatrix}$, $\boldsymbol{c}_2=\begin{bmatrix}-5\\7\end{bmatrix}$, and

consider the bases for $\boldsymbol{R}^2$ given by $\beta = \{\boldsymbol{b}_1, \boldsymbol{b}_2\}$ and $\boldsymbol{C} = \{\boldsymbol{c}_1, \boldsymbol{c}_2\}$.

(1) Find the change-of-coordinates matrix from $\boldsymbol{C}$ to β .

(2) Find the change-of-coordinates matrix from β to $\boldsymbol{C}$.

Solution.

(1) $$\underset{\boldsymbol{\beta}\leftarrow C}{\boldsymbol{P}}=\underset{\boldsymbol{\beta}\leftarrow E}{\boldsymbol{P}}\ \underset{E\leftarrow C}{\boldsymbol{P}}=\left(\underset{E\leftarrow\boldsymbol{\beta}}{\boldsymbol{P}}\right)^{-1}\underset{E\leftarrow C}{\boldsymbol{P}}=\begin{bmatrix}5&3\\6&4\end{bmatrix}$$

$$\left[\underset{E\leftarrow\boldsymbol{\beta}}{\boldsymbol{P}}\ \vdots\ \underset{E\leftarrow C}{\boldsymbol{P}}\right]\rightarrow\left[\boldsymbol{I}\ \vdots\ \left(\underset{E\leftarrow\boldsymbol{\beta}}{\boldsymbol{P}}\right)^{-1}\underset{E\leftarrow C}{\boldsymbol{P}}\right]$$

$$\begin{bmatrix}1&-2&:&-7&-5\\-3&4&:&9&7\end{bmatrix}\rightarrow\begin{bmatrix}1&0&:&5&3\\0&1&:&6&4\end{bmatrix}.$$

$$(2)\quad \underset{C\leftarrow\boldsymbol{\beta}}{\boldsymbol{P}} = \underset{C\leftarrow E}{\boldsymbol{P}}\ \underset{E\leftarrow\boldsymbol{\beta}}{\boldsymbol{P}} = \left(\underset{E\leftarrow C}{\boldsymbol{P}}\right)^{-1} \underset{E\leftarrow\boldsymbol{\beta}}{\boldsymbol{P}} = \begin{bmatrix} 2 & -3/2 \\ -3 & 5/2 \end{bmatrix}$$

$$\left[\underset{E\leftarrow C}{\boldsymbol{P}} \;\vdots\; \underset{E\leftarrow\boldsymbol{\beta}}{\boldsymbol{P}}\right] \rightarrow \left[\boldsymbol{I} \;\vdots\; \left(\underset{E\leftarrow C}{\boldsymbol{P}}\right)^{-1} \underset{E\leftarrow\boldsymbol{\beta}}{\boldsymbol{P}}\right]$$

$$\begin{bmatrix} -7 & -5 & : & 1 & -2 \\ 9 & 7 & : & -3 & 4 \end{bmatrix} \rightarrow \begin{bmatrix} 1 & 0 & : & 2 & -3/2 \\ 0 & 1 & : & -3 & 5/2 \end{bmatrix}.$$

- Exercises for Section 4.7.

- **Blind-spot detection** (Demonstration)

  Demonstrate 8 scenarios (1 min. 38 sec.)

  Traffic sign on ground
  Tree shadow
  Cast shadow
  Rainy day
  Same-speed car
  Night (white car)
  Night (light)
  Night (dark)