3. Determinants

- Determinants are tools for analytic geometry and other parts of mathematics.
  For example,
  (1) provide an invertibility criterion for a square matrix.
  (2) give formulas for $\boldsymbol{A}^{-1}$ and $\boldsymbol{A}^{-1}\boldsymbol{b}$.
  (3) derive the geometric interpretation of determinants.

3.1 Introduction of determinants

- Notation
  Assume that $\boldsymbol{A}$ is a square matrix. Let $\boldsymbol{A}_{ij}$ denote the submatrix formed by deleting the *i* th row and *j* th column of $\boldsymbol{A}$.

- Definition
  For $n \geq 2$, the determinant of an nxn matrix $\boldsymbol{A} = [a_{ij}]$ is the sum of n terms of the form $\pm a_{ij}\, det\boldsymbol{A}_{ij}$, with plus and minus signs alternative. That is

$$\det \boldsymbol{A} = \sum_{j=1}^{n}(-1)^{1+j}a_{1j}\det \boldsymbol{A}_{1j}$$

$$= a_{11}\det \boldsymbol{A}_{11} - a_{12}\det \boldsymbol{A}_{12} + \ldots + (-1)^{1+n}a_{1n}\det \boldsymbol{A}_{1n}$$

- Ex.1.

$$\boldsymbol{A} = \begin{bmatrix} 1 & 5 & 0 \\ 2 & 4 & -1 \\ 0 & -2 & 0 \end{bmatrix}$$

$$det\, A = a_{11}det\, \boldsymbol{A}_{11} - a_{12}det\, \boldsymbol{A}_{12} + a_{13}det\, \boldsymbol{A}_{13}$$

$$= 1\, det\begin{bmatrix} 4 & -1 \\ -2 & 0 \end{bmatrix} - 5\, det\begin{bmatrix} 2 & -1 \\ 0 & 0 \end{bmatrix} + 0\, det\begin{bmatrix} 2 & 4 \\ 0 & -2 \end{bmatrix}$$

$$= -2 - 0 - 0$$

$$= -2.$$

- Another notation $det\, \boldsymbol{A} = |\, \boldsymbol{A}\, |$.

- For $n \leq 3$, the determinant can be computed by

$$\boldsymbol{A} = \begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{bmatrix}$$

$$det\, \boldsymbol{A} = \left\| \begin{matrix} a_{11} & . & . \\ . & a_{22} & . \\ . & . & a_{33} \end{matrix} \right\| + \left\| \begin{matrix} . & a_{12} & . \\ . & . & a_{23} \\ a_{31} & . & . \end{matrix} \right\| + \left\| \begin{matrix} . & . & a_{13} \\ a_{21} & . & . \\ . & a_{32} & . \end{matrix} \right\|$$

$$- \left\| \begin{matrix} . & . & a_{13} \\ . & a_{22} & . \\ a_{31} & . & . \end{matrix} \right\| - \left\| \begin{matrix} a_{11} & . & . \\ . & . & a_{23} \\ . & a_{32} & . \end{matrix} \right\| - \left\| \begin{matrix} . & a_{12} & . \\ a_{21} & . & . \\ . & . & a_{33} \end{matrix} \right\|.$$

- Note
  The formula can only be used for the cases of $n \leq 3$.
  Why the formula can not be used for the cases of $n > 3$?

- Example,

$$A = \begin{bmatrix} a_{11} & a_{12} & a_{13} & a_{14} \\ a_{21} & a_{22} & a_{23} & a_{24} \\ a_{31} & a_{32} & a_{33} & a_{34} \\ a_{41} & a_{42} & a_{43} & a_{44} \end{bmatrix}, \text{ if we take the special formula, } det\ A =$$

$$\begin{bmatrix} a_{11} & \cdot & \cdot & \cdot \\ \cdot & a_{22} & \cdot & \cdot \\ \cdot & \cdot & a_{33} & \cdot \\ \cdot & \cdot & \cdot & a_{44} \end{bmatrix} + \begin{bmatrix} \cdot & a_{12} & \cdot & \cdot \\ \cdot & \cdot & a_{23} & \cdot \\ \cdot & \cdot & \cdot & a_{34} \\ a_{41} & \cdot & \cdot & \cdot \end{bmatrix} + \begin{bmatrix} \cdot & \cdot & a_{13} & \cdot \\ \cdot & \cdot & \cdot & a_{24} \\ a_{31} & \cdot & \cdot & \cdot \\ \cdot & a_{42} & \cdot & \cdot \end{bmatrix} + \begin{bmatrix} \cdot & \cdot & \cdot & a_{14} \\ a_{21} & \cdot & \cdot & \cdot \\ \cdot & a_{32} & \cdot & \cdot \\ \cdot & \cdot & a_{43} & \cdot \end{bmatrix}$$

$$- \begin{bmatrix} \cdot & \cdot & \cdot & a_{14} \\ \cdot & \cdot & a_{23} & \cdot \\ \cdot & a_{32} & \cdot & \cdot \\ a_{41} & \cdot & \cdot & \cdot \end{bmatrix} - \begin{bmatrix} a_{11} & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & a_{24} \\ \cdot & \cdot & a_{33} & \cdot \\ \cdot & a_{42} & \cdot & \cdot \end{bmatrix} - \begin{bmatrix} \cdot & a_{12} & \cdot & \cdot \\ a_{21} & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & a_{34} \\ \cdot & \cdot & a_{43} & \cdot \end{bmatrix} - \begin{bmatrix} \cdot & \cdot & a_{13} & \cdot \\ \cdot & a_{22} & \cdot & \cdot \\ a_{31} & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & a_{44} \end{bmatrix}.$$

It only contains partial terms; not complete.

- Definition

Given $\boldsymbol{A} = [a_{ij}]$, the (i,j)-cofactor of $\boldsymbol{A}$ is the number c_{ij} given by $c_{ij} = (-1)^{i+j}\ det\ \boldsymbol{A}_{ij}$.

- According to the definition of cofactor

$$det\ \boldsymbol{A} = a_{11}c_{11} + a_{12}c_{12} + \ldots + a_{1n}c_{1n}$$

The formula is called the cofactor expansion along the first row.

- **Theorem 1**

The determinant of $\boldsymbol{A}_{nxn}$ may be computed by a cofactor expansion along any row or down any column. The expansion across the i th row is given

$$det\ \boldsymbol{A} = a_{i1}c_{i1} + a_{i2}c_{i2} + \ldots + a_{in}c_{in}\ .$$

The expansion across the j th column is given

$$det\ \boldsymbol{A} = a_{1j}\,c_{1j} + a_{2j}\,c_{2j} + \ldots + a_{nj}\,c_{nj}\ .$$

- Theorem 1 is helpful for computing the determinant of a matrix that contains many zeros. The expansion is done across the row or column with most zeros.

- Ex.3.

$$\begin{vmatrix} 3 & -7 & 8 & 9 & -6 \\ 0 & 2 & -5 & 7 & 3 \\ 0 & 0 & 1 & 5 & 0 \\ 0 & 0 & 2 & 4 & -1 \\ 0 & 0 & 0 & -2 & 0 \end{vmatrix}$$

$$= 3\begin{vmatrix} 2 & -5 & 7 & 3 \\ 0 & 1 & 5 & 0 \\ 0 & 2 & 4 & -1 \\ 0 & 0 & -2 & 0 \end{vmatrix}$$

$$= 3 \cdot 2\begin{vmatrix} 1 & 5 & 0 \\ 2 & 4 & -1 \\ 0 & -2 & 0 \end{vmatrix}$$

$$= 3 \cdot 2 \cdot (-2) \cdot (-1)^{3+2} \cdot \begin{vmatrix} 1 & 0 \\ 2 & -1 \end{vmatrix}$$

$$= 3 \cdot 2 \cdot (-2) \cdot (-1) \cdot (-1)$$

$$= -12.$$

- Theorem 2

  If ***A*** is a triangular matrix, then *det* ***A*** is the product of the entries on the main diagonal of ***A***.

- Exercise of Section 3.1.

3.2 Properties of determinants

- Theorem 3 (row operations)
  Let ***A*** be a square matrix.

  (*a*) replacement

  A multiplication of one row of ***A*** is added to another to produce a matrix ***B***, then *det* ***B*** = *det* ***A***.

  $E_r \Rightarrow det\ E_r = 1.$

  (*b*) interchange

  Two rows of ***A*** are interchanged to produce ***B***, then *det* ***B*** = - *det* ***A***.

  $E_i \Rightarrow det\ E_i = -1.$

  (*c*) scaling

  One row of ***A*** is multipled by *k* to produce ***B***, then *det* ***B*** = *k det* ***A***. $E_s \Rightarrow det\ E_s = k.$

- Ex.2.

$$\begin{vmatrix} 2 & -8 & 6 & 8 \\ 3 & -9 & 5 & 10 \\ -3 & 0 & 1 & -2 \\ 1 & -4 & 0 & 6 \end{vmatrix}$$

$$= 2 \cdot \begin{vmatrix} 1 & -4 & 3 & 4 \\ 3 & -9 & 5 & 10 \\ -3 & 0 & 1 & -2 \\ 1 & -4 & 0 & 6 \end{vmatrix} = 2 \cdot \begin{vmatrix} 1 & -4 & 3 & 4 \\ 0 & 3 & -4 & -2 \\ 0 & -12 & 10 & 10 \\ 0 & 0 & -3 & 2 \end{vmatrix}$$

$$= 2 \cdot \begin{vmatrix} 1 & -4 & 3 & 4 \\ 0 & 3 & -4 & -2 \\ 0 & 0 & -6 & 2 \\ 0 & 0 & -3 & 2 \end{vmatrix} = 2 \cdot \begin{vmatrix} 1 & -4 & 3 & 4 \\ 0 & 3 & -4 & -2 \\ 0 & 0 & -6 & 2 \\ 0 & 0 & 0 & 1 \end{vmatrix}$$

$$= 2\,(1)\,(3)\,(-6)\,(1)$$

$$= -36.$$

Proof of Theorem 3

By induction.

Assume $\boldsymbol{A}$ is an nxn matrix and $\boldsymbol{E}$ is an nxn elementary matrix. To prove that n = 2, it is true.

$$\begin{vmatrix} a & b \\ a+c & b+d \end{vmatrix} = \begin{vmatrix} a & b \\ a & b \end{vmatrix} + \begin{vmatrix} a & b \\ c & d \end{vmatrix} = \begin{vmatrix} a & b \\ c & d \end{vmatrix}$$

$$\begin{vmatrix} a & b \\ c & d \end{vmatrix} = ad - bc = -(bc - ad) = -\begin{vmatrix} c & d \\ a & b \end{vmatrix}$$

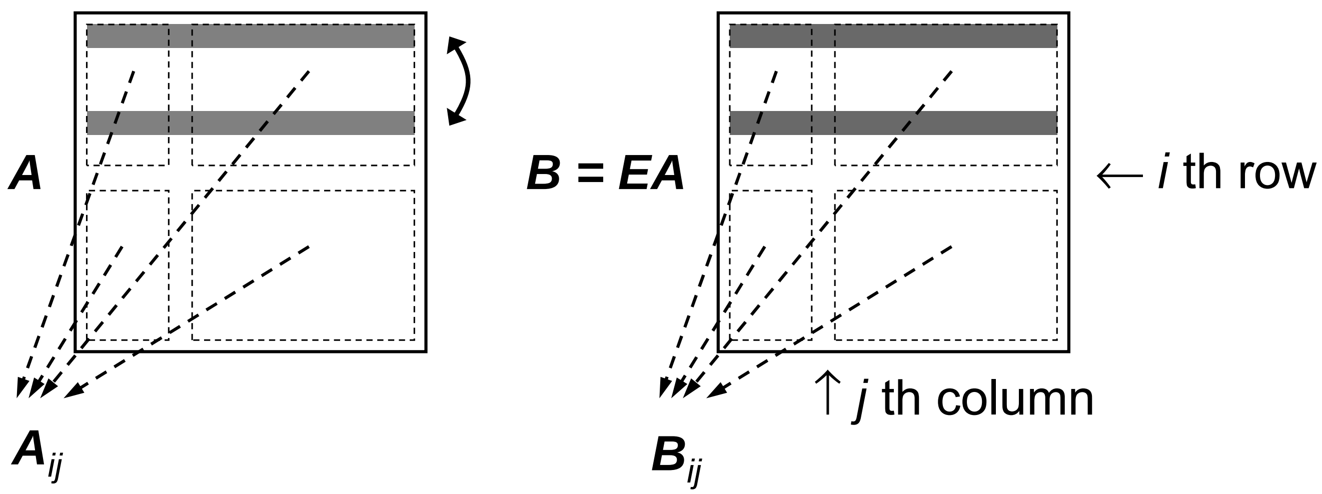
$$\begin{vmatrix} a & b \\ kc & kd \end{vmatrix} = kad - kbc = k(ad - bc) = k\begin{vmatrix} a & b \\ c & d \end{vmatrix}$$

Assume that $n = m$, it is true.

To prove that $n = m+1$, it is true. $\boldsymbol{EA} = \boldsymbol{B}$. The action of $\boldsymbol{E}$ on $\boldsymbol{A}$ involves either two rows or only one row. So we may expand *det* $\boldsymbol{EA}$ along the non-involving row, say, row i.

Let $\boldsymbol{A}_{ij}$ be the matrix obtained by deleting row i and column j from $\boldsymbol{A}$. Then the rows of $\boldsymbol{B}_{ij}$ are obtained from the rows of $\boldsymbol{A}_{ij}$ by the same type of elementary row operation as $\boldsymbol{E}$ performs on $\boldsymbol{A}$ and then $\boldsymbol{B}_{ij} = \boldsymbol{E}\,\boldsymbol{A}_{ij} \Rightarrow det\,\boldsymbol{B}_{ij} = \alpha\, det\,\boldsymbol{A}_{ij}$, where α = 1, -1 or k.

$$\underset{(m+1)\times(m+1)}{\boldsymbol{E}\,\boldsymbol{A} = \boldsymbol{B}} \Rightarrow \underset{m\times m}{\boldsymbol{B}_{ij} = \boldsymbol{E}\,\boldsymbol{A}_{ij}} \Rightarrow det\,\boldsymbol{B}_{ij} = \alpha\, det\,\boldsymbol{A}_{ij}$$



$$\begin{aligned} det\,\boldsymbol{B} &= \sum_{j=1}^{n} b_{ij}(-1)^{i+j}\,det\,\boldsymbol{B}_{ij} \\ &= \sum_{j=1}^{n} a_{ij}(-1)^{i+j}\;\alpha\,det\,\boldsymbol{A}_{ij} \\ &= \alpha\sum_{j=1}^{n} a_{ij}(-1)^{i+j}\,det\,\boldsymbol{A}_{ij} \\ &= \alpha\,det\,\boldsymbol{A}. \end{aligned}$$

- Theorem 4
  A square matrix ***A*** is invertible if and only if *det* ***A*** ≠ 0.

- Corollary
  det $\boldsymbol{A}_{nxn}$ = 0 if and only if the rows or columns of ***A*** are linearly dependent.

- Ex.3.

$$\begin{vmatrix} 3 & -1 & 2 & -5 \\ 0 & 5 & -3 & -6 \\ -6 & 7 & -7 & 4 \\ -5 & -8 & 0 & 9 \end{vmatrix} = \begin{vmatrix} 3 & -1 & 2 & -5 \\ 0 & 5 & -3 & -6 \\ 0 & 5 & -3 & -6 \\ -5 & -8 & 0 & 9 \end{vmatrix} = 0.$$

- Ex.4.

$$\begin{vmatrix} 0 & 1 & 2 & -1 \\ 2 & 5 & -7 & 3 \\ 0 & 3 & 6 & 2 \\ -2 & -5 & 4 & -2 \end{vmatrix} = \begin{vmatrix} 0 & 1 & 2 & -1 \\ 2 & 5 & -7 & 3 \\ 0 & 3 & 6 & 2 \\ 0 & 0 & -3 & 1 \end{vmatrix} = 2\,(-1)^{2+1}\begin{vmatrix} 1 & 2 & -1 \\ 3 & 6 & 2 \\ 0 & -3 & 1 \end{vmatrix}$$

$$= -2\begin{vmatrix} 1 & 2 & -1 \\ 0 & 0 & 5 \\ 0 & -3 & 1 \end{vmatrix} = -2\,(-1)^{1+1}\,1\begin{vmatrix} 0 & 5 \\ -3 & 1 \end{vmatrix} = -2\,(15) = -30.$$

Column operations

- Theorem 5

  $\boldsymbol{A}_{nxn}$ is a square matrix. $det\ \boldsymbol{A}^T = det\ \boldsymbol{A}$.

  proof.

  By induction. If $n = 2$, trivial.

  Let $n = m$ be true, to show $n = m+1$ is true.

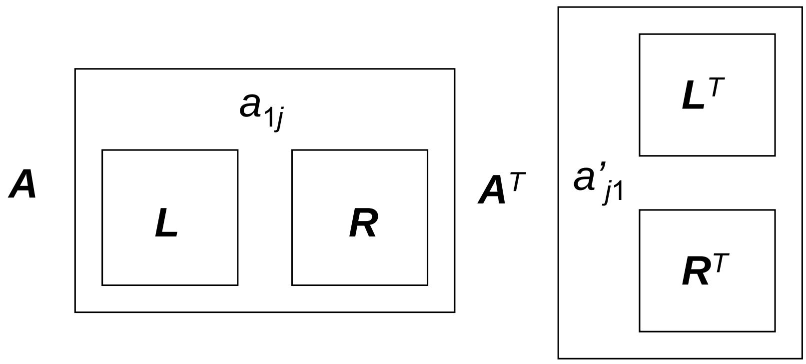
  The cofactor of a_{1j} in $\boldsymbol{A}$ ($c_{1j} = (-1)^{1+j}\ det\ \boldsymbol{A}_{1j}$)

  = The cofactor of a'_{j1} in $\boldsymbol{A}^T$ ($c'_{j1} = (-1)^{j+1} det\ \boldsymbol{A}^T{}_{1j}$)

$$det\ \boldsymbol{A} = \sum_{j=1}^{n} a_{1j}\, c_{1j}$$

$$det\ \boldsymbol{A}^T = \sum_{j=1}^{n} a'_{j1}\, c'_{j1}$$

$$\Rightarrow det\ \boldsymbol{A} = det\ \boldsymbol{A}^T,$$



since $a_{1j} = a'_{j1}$ and $c_{1j} = c'_{j1}$ for j =1, 2, …, n.

Matrix products

- Theorem 6

  If $\boldsymbol{A}$ and $\boldsymbol{B}$ are nxn matrices, then $det\,\boldsymbol{AB} = (det\boldsymbol{A})\ (det\boldsymbol{B})$.

  proof.

  If $\boldsymbol{A}$ is not invertible, then neither is $\boldsymbol{AB}$.

  (If $\boldsymbol{AB}$ is invertible, we take $\boldsymbol{C} = \boldsymbol{B}\ (\boldsymbol{AB})^{-1}$; then

  $\boldsymbol{AC} = \boldsymbol{AB}\ (\boldsymbol{AB})^{-1} = \boldsymbol{I} \Rightarrow \boldsymbol{A}$ is invertible.)

  In this case, $(det\boldsymbol{A})\ (det\boldsymbol{B}) = 0 = det\ \boldsymbol{A}\ \boldsymbol{B}$.

  If $\boldsymbol{A}$ is invertible, $\boldsymbol{A}$ is a product of elementary matrices,

  $\boldsymbol{A} = \boldsymbol{E}_p\boldsymbol{E}_{p-1}\ldots\boldsymbol{E}_1$.

  $|\boldsymbol{AB}| = |\boldsymbol{E}_p\boldsymbol{E}_{p-1}\ldots\boldsymbol{E}_1\boldsymbol{B}| = |\boldsymbol{E}_p||\boldsymbol{E}_{p-1}\ldots\boldsymbol{E}_1\boldsymbol{B}| = \ldots$

  $= |\boldsymbol{E}_p||\boldsymbol{E}_{p-1}|\ldots|\boldsymbol{E}_1||\boldsymbol{B}| = |\boldsymbol{E}_p\boldsymbol{E}_{p-1}\ldots\boldsymbol{E}_1||\boldsymbol{B}|$

  $= |\boldsymbol{A}||\boldsymbol{B}|$.

- Note that $det(\boldsymbol{A}+\boldsymbol{B}) \neq det\,\boldsymbol{A} + det\,\boldsymbol{B}$.

  For example, $\begin{vmatrix} 1+5 & 2+6 \\ 3+7 & 4+8 \end{vmatrix} \neq \begin{vmatrix} 1 & 2 \\ 3 & 4 \end{vmatrix} + \begin{vmatrix} 5 & 6 \\ 7 & 8 \end{vmatrix}$.

A linearity property of the determinant function

- Assume that $\boldsymbol{A} = [\boldsymbol{a}_1\ \ \boldsymbol{a}_2 \dots \boldsymbol{a}_n]$. $det\,\boldsymbol{A} = f(a_1, \dots, a_n)$, f is a linear function. Suppose that the j th column of $\boldsymbol{A}$ is allowed to vary, and other column are held fixed. We write

  $\boldsymbol{A} = [\boldsymbol{a}_1\ \ \boldsymbol{a}_2 \dots \boldsymbol{a}_{j\text{-}1}\ \ \boldsymbol{x}\ \ \boldsymbol{a}_{j+1} \dots\ \ \boldsymbol{a}_n\,]$.

  Define a transformation $\boldsymbol{T}$ from $\boldsymbol{R}^{\,n}$ to $\boldsymbol{R}$ by

  $\boldsymbol{T}(\boldsymbol{x}) = det\,[\boldsymbol{a}_1\ \ \boldsymbol{a}_2 \dots \boldsymbol{a}_{j\text{-}1}\ \ \boldsymbol{x}\ \ \boldsymbol{a}_{j+1} \dots\ \ \boldsymbol{a}_n\,]$. $\leftarrow$ (new definition)

  Then $\boldsymbol{T}(k\boldsymbol{x}) = k\boldsymbol{T}(\boldsymbol{x})$ for all scalars k and all $\boldsymbol{x}$ in $\boldsymbol{R}^{\,n}$.

  $\boldsymbol{T}(\boldsymbol{u} + \boldsymbol{v}) = \boldsymbol{T}(\boldsymbol{u}) + \boldsymbol{T}(\boldsymbol{v})$ for all $\boldsymbol{u}$, $\boldsymbol{v}$ in $\boldsymbol{R}^{\,n}$.

- Exercises of Section 3.2.

3.3 Cramer's rule, volume, and linear transformations

- Notation

  For any nxn matrix $\boldsymbol{A}$ and $\boldsymbol{b}$ in $\boldsymbol{R}^{\,n}$. Let $\boldsymbol{A}_i(\boldsymbol{b})$ be the matrix obtained from $\boldsymbol{A}$ by replacing column i by the vector $\boldsymbol{b}$,

  $\boldsymbol{A}_i(\boldsymbol{b}) = [\boldsymbol{a}_1\ \boldsymbol{a}_2 \dots \boldsymbol{b} \dots \boldsymbol{a}_n\,]$.

- Theorem 7 (Cramer's rule)

  $\boldsymbol{A}_{n\text{x}n}$ is invertible, for any $\boldsymbol{b}$ in $\boldsymbol{R}^{\,n}$, then the unique solution $\boldsymbol{x}$ of $\boldsymbol{Ax} = \boldsymbol{b}$ has entries $x_i = \dfrac{det\ \boldsymbol{A}_i(\boldsymbol{b})}{det\ \boldsymbol{A}},\ \ i = 1, 2, \dots, n.$

  Proof.

  Denote $\boldsymbol{A} = [\boldsymbol{a}_1\ \boldsymbol{a}_2 \dots \boldsymbol{a}_n\,]$ and $\boldsymbol{I} = [\boldsymbol{e}_1\ \boldsymbol{e}_2 \dots \boldsymbol{e}_n\,]$

  $\boldsymbol{A}\ \boldsymbol{I}_i(\boldsymbol{x}) = \boldsymbol{A}\ [\boldsymbol{e}_1\ \boldsymbol{e}_2 \,..\, \boldsymbol{x} \,..\, \boldsymbol{e}_n\,] = [\boldsymbol{Ae}_1\ \boldsymbol{Ae}_2 \,..\, \boldsymbol{Ax} \,..\, \boldsymbol{Ae}_n\,]$

  $= [\boldsymbol{a}_1\ \boldsymbol{a}_2 \,..\, \boldsymbol{b} \,..\, \boldsymbol{a}_n\,] = \boldsymbol{A}_i(\boldsymbol{b})$

  By Theorem 6 ($det\,\boldsymbol{AB} = det\,\boldsymbol{A}\ det\,\boldsymbol{B}$)

  $(det\,\boldsymbol{A})\ (det\,\boldsymbol{I}_i(\boldsymbol{x})) = det\,\boldsymbol{A}_i(\boldsymbol{b}) \Rightarrow (det\,\boldsymbol{A})\ x_i = det\,\boldsymbol{A}_i(\boldsymbol{b}) \Rightarrow$ $x_i = \dfrac{det\ \boldsymbol{A}_i(\boldsymbol{b})}{det\ \boldsymbol{A}}$.

- Ex.2.

$$\begin{cases} 3sx_1 - 2x_2 = 4 \\ -6x_1 + sx_2 = 1 \end{cases}$$ has a unique solution.

to find *s* and the solution.

Answer.

$det\ \boldsymbol{A} \neq 0 \Rightarrow$ exist a unique solution.

$$det\ \boldsymbol{A} = \begin{vmatrix} 3s & -2 \\ -6 & s \end{vmatrix} = 3s^2 - 12 = 3(s+2)(s-2)$$

$$det\ \boldsymbol{A} \neq 0 \Rightarrow s \neq \pm 2$$

$$x_1 = \frac{4s+2}{3(s+2)(s-2)}$$

$$x_2 = \frac{3s+24}{3(s+2)(s-2)} = \frac{s+8}{(s+2)(s-2)}.$$

A formula for $\boldsymbol{A}^{-1}$ (the 3rd method)

- Theorem 8

$\boldsymbol{A}_{nxn}$ is invertible, then $\boldsymbol{A}^{-1} = \dfrac{1}{det\ \boldsymbol{A}} adj\ \boldsymbol{A}$

where $adj\ \boldsymbol{A} = \begin{bmatrix} c_{11} & c_{21} & .. & c_{n1} \\ c_{12} & c_{22} & .. & c_{n2} \\ \vdots & \vdots & . & \vdots \\ c_{1n} & c_{2n} & .. & c_{nn} \end{bmatrix}$ is called the adjugate of $\boldsymbol{A}$

and c_{ij} is the (i,j)-cofactor of $\boldsymbol{A}$.

> $\boldsymbol{A}\,\boldsymbol{A}^{-1} = \boldsymbol{I} = [\mathbf{e}_1\ \mathbf{e}_2\ ..\ \mathbf{e}_n]$
> $\boldsymbol{A}^{-1} = [\boldsymbol{x}_1\ \boldsymbol{x}_2\ ..\ \boldsymbol{x}_n]$
> $\boldsymbol{A}\,\boldsymbol{x}_j = \mathbf{e}_j$

Proof.

By Cramer's rule, the *j* th column of $\boldsymbol{A}^{-1}$ is a vector $\boldsymbol{x}_j$ that satisfies $\boldsymbol{A}\boldsymbol{x}_j = \mathbf{e}_j$ and the (i, j) entry of $\boldsymbol{A}^{-1}$ is

$$x_{ij} = \frac{det\ \boldsymbol{A}_i(\mathbf{e}_j)}{det\ \boldsymbol{A}}.$$

row column

$$det\, \boldsymbol{A}_i(e_j) = \begin{vmatrix} x & x & .. & 0 & .. & x \\ & & & \vdots & & \\ & & & 0 & & \\ \cdots & \cdots & \cdots & 1 & \cdots & \cdots \\ & & & 0 & & \\ & & & 0 & & \end{vmatrix} = (-1)^{j+i}\, det\, \boldsymbol{A}_{ji} = c_{ji}$$

the j th row (the row containing 1)

↑ the i th column

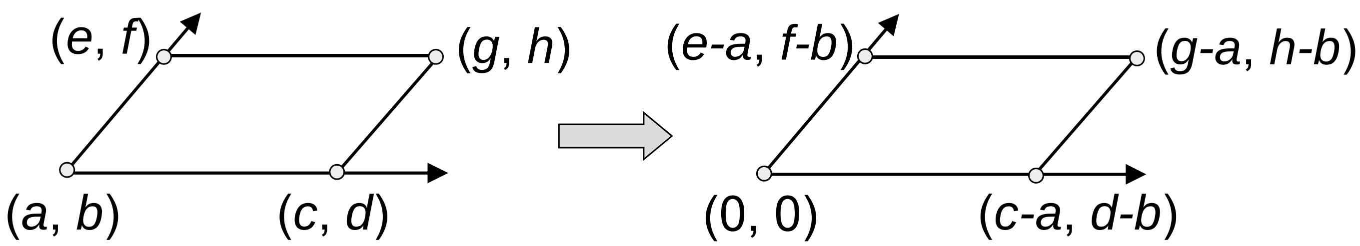
Thus

$$\boldsymbol{A}^{-1} = \frac{1}{det\, \boldsymbol{A}} \begin{bmatrix} c_{11} & c_{21} & .. & c_{n1} \\ c_{12} & c_{22} & .. & c_{n2} \\ \vdots & \vdots & \vdots & \vdots \\ c_{1n} & c_{2n} & .. & c_{nn} \end{bmatrix}.$$

- Examples. omitted.

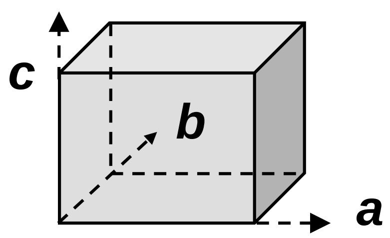
Determinants as area or volume

- 2-dimensional cases



$$Area = \left| det \begin{bmatrix} c-a & e-a \\ d-b & f-b \end{bmatrix} \right|$$

- 3-dimensional cases



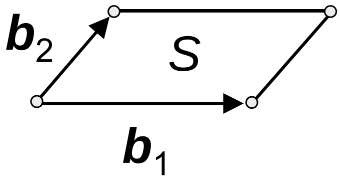
$$Volume = \left| det\, [\boldsymbol{a} \quad \boldsymbol{b} \quad \boldsymbol{c}] \right|$$

Linear transformation

- Theorem 10

  (a) Let $\boldsymbol{T}: \boldsymbol{R}^2 \to \boldsymbol{R}^2$ be the linear transformation determined by a 2×2 matrix $\boldsymbol{A}$. If S is a parallelogram in $\boldsymbol{R}^2$, then {area of $\boldsymbol{T}(S)$} = $|det\ \boldsymbol{A}|$ {area of S}.

  (b) If $\boldsymbol{T}: \boldsymbol{R}^3 \to \boldsymbol{R}^3$ determined by a 3×3 matrix $\boldsymbol{A}$ and S is a parallelepiped in $\boldsymbol{R}^3$, then {volume of $\boldsymbol{T}(S)$} = $|det\ \boldsymbol{A}|$ {volume of S}.



  Proof.

  (a) $\boldsymbol{A} = [\boldsymbol{a}_1\ \boldsymbol{a}_2]$

  $S = \{ s_1\boldsymbol{b}_1 + s_2\boldsymbol{b}_2 : 0 \le s_1 \le 1, 0 \le s_2 \le 1 \}$

  area of $S = |\ det\ [\boldsymbol{b}_1\ \boldsymbol{b}_2]\ |$

  $\boldsymbol{T}(S) = \boldsymbol{T}(s_1\boldsymbol{b}_1 + s_2\boldsymbol{b}_2) = s_1\boldsymbol{T}(\boldsymbol{b}_1) + s_2\boldsymbol{T}(\boldsymbol{b}_2) = s_1\boldsymbol{A}\boldsymbol{b}_1 + s_2\boldsymbol{A}\boldsymbol{b}_2$ is the parallelogram determined by the columns of matrix $[\boldsymbol{A}\boldsymbol{b}_1\ \boldsymbol{A}\boldsymbol{b}_2]$. $[\boldsymbol{A}\boldsymbol{b}_1\ \ \boldsymbol{A}\boldsymbol{b}_2] = \boldsymbol{A}[\boldsymbol{b}_1\ \ \boldsymbol{b}_2] = \boldsymbol{A}\boldsymbol{B} \ \Rightarrow\ |\boldsymbol{A}\boldsymbol{b}_1\ \ \boldsymbol{A}\boldsymbol{b}_2| = |\boldsymbol{A}||\boldsymbol{B}|$

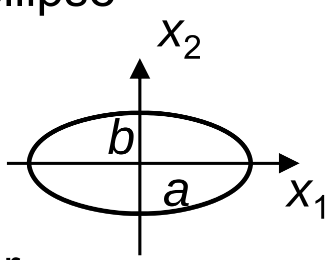
  {area of $T(s)$} = $|\,det\ \boldsymbol{A}\,|$ {area of $\boldsymbol{S}$}.

- The conclusion of Theorem 10 hold whenever $\boldsymbol{S}$ is a region in $\boldsymbol{R}^2$ with finite area or a region in $\boldsymbol{R}^3$ with finite volume.

- Ex. 5.

  Find the area of the region $\boldsymbol{E}$ bounded by the ellipse

  $$\frac{x_1^2}{a^2} + \frac{x_2^2}{b^2} = 1, \quad a > 0,\ b > 0.$$



  Answer.

  $\boldsymbol{E}$ is an image of the unit disk D under the linear transformation $\boldsymbol{T}$ determined by $\boldsymbol{A} = \begin{bmatrix} a & 0 \\ 0 & b \end{bmatrix}$

  {area of ellipse} = {are of $\boldsymbol{T}(D)$} = $|det\,\boldsymbol{A}|$ {area of D}

  $= ab\,\pi\, 1^2 = ab\,\pi.$

- Exercise of Section 3.3.