6. Laplace Transforms

6.1 Laplace transform, inverse transform, linearity

- Laplace transform

$$F(s) = \int_0^{\infty} e^{-st} f(t)\,dt$$

of the function $f(t)$, and be denoted by $\mathscr{L}(f)$.

- Inverse Laplace transform

$f(t) = \mathscr{L}^{-1}(F)$.

- EX.1.

$f(t) = 1$, when $t \geq 0$.

$$\mathscr{L}[f(t)] = \mathscr{L}(1) = \int_0^{\infty} e^{-st} dt = -\frac{1}{s} e^{-st}\Big|_0^{\infty} = \frac{1}{s}, \text{ when } s > 0.$$

- The notation

$$\int_0^\infty e^{-st} f(t)dt = \lim_{T\to\infty} \int_0^T e^{-st} f(t)dt .$$

- EX.2.

$$\mathscr{L}(e^{at}) = \int_0^\infty e^{-st} e^{at} dt = \frac{1}{a-s} e^{-(s-a)t}\Big|_0^\infty = \frac{1}{s-a}, \text{ when } s-a>0 .$$

- Sometimes we may obtain the Laplace transform of a function indirectly from the definition.

- Theorem 1 (Linearity of the Laplace transform)

  The Laplace transform is a linear operation; that is, for any functions $f(t)$ and $g(t)$ whose Laplace transforms exist and any constant a and b,

  $\mathscr{L}\{af(t) + bg(t)\} = a\,\mathscr{L}\{f(t)\} + b\,\mathscr{L}\{g(t)\}$

  $\mathscr{L}^{-1}\{aF(s) + bG(s)\} = a\,\mathscr{L}^{-1}\{F(s)\} + b\,\mathscr{L}^{-1}\{G(s)\}$.

- EX.3.

$$\mathscr{L}(\cosh at) = \mathscr{L}\left(\frac{e^{at}+e^{-at}}{2}\right) = \frac{1}{2}\mathscr{L}(e^{at}) + \frac{1}{2}\mathscr{L}(e^{-at}) = \frac{1}{2}\frac{1}{s-a} + \frac{1}{2}\frac{1}{s+a} = \frac{s}{s^2-a^2}.$$

- EX.

$$\mathscr{L}^{-1}\left(\frac{1}{(s-a)(s-b)}\right) = \mathscr{L}^{-1}\left\{\frac{1}{a-b}\left(\frac{1}{s-a} - \frac{1}{s-b}\right)\right\}$$

$$= \frac{1}{a-b}\left[\mathscr{L}^{-1}\left(\frac{1}{s-a}\right) - \mathscr{L}^{-1}\left(\frac{1}{s-b}\right)\right] = \frac{1}{a-b}\left[e^{at} - e^{bt}\right].$$

- EX.

$$\mathscr{L}(e^{i\omega t}) = \frac{1}{s-i\omega} = \frac{s+i\omega}{(s-i\omega)(s+i\omega)} = \frac{s+i\omega}{s^2+\omega^2} = \frac{s}{s^2+\omega^2} + i\frac{\omega}{s^2+\omega^2}.$$

❁ Ex.4.

Prove that $\mathscr{L}(\cos\omega t) = \dfrac{s}{s^2+\omega^2}$ and

$$\mathscr{L}(\sin\omega t) = \frac{\omega}{s^2+\omega^2}.$$

since $\mathscr{L}(e^{at}) = \dfrac{1}{s-a}$

$$\mathscr{L}(e^{i\omega t}) = \frac{1}{s-i\omega} = \frac{s+i\omega}{(s-i\omega)(s+i\omega)} = \frac{s+i\omega}{s^2+\omega^2} = \frac{s}{s^2+\omega^2} + i\frac{\omega}{s^2+\omega^2}.$$

By Euler formula: $e^{i\omega t} = \cos\omega t + i\sin\omega t$, we have

$$\mathscr{L}(e^{i\omega t}) = \mathscr{L}(\cos\omega t + i\sin\omega t) = \mathscr{L}(\cos\omega t) + i\,\mathscr{L}(\sin\omega t).$$

Thus $\boxed{\mathscr{L}[\cos\omega t] = \dfrac{s}{s^2+\omega^2}}$

$$\boxed{\mathscr{L}[\sin\omega t] = \frac{\omega}{s^2+\omega^2}.}$$

First shifting theorem

❁ Theorem 2 (First shifting theorem)

If $f(t)$ has the transform $F(s)$ (where $s > k$), then $e^{at} f(t)$ has the transform $F(s - a)$ (where $s - a > k$). In formulas,

$\boxed{\mathscr{L}\{e^{at} f(t)\} = F(s-a)}$ and

$e^{at} f(t) = \mathscr{L}^{-1}\{F(s-a)\}$.

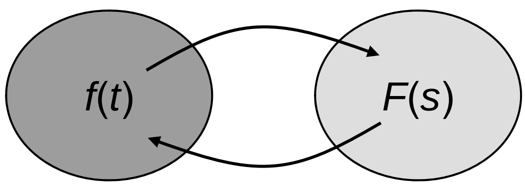
Proof

By definition $F(s) = \int_0^\infty e^{-st} f(t)\,dt$,

thus $F(s-a) = \int_0^\infty e^{-(s-a)t} f(t)\,dt = \int_0^\infty e^{-st}\left[e^{at} f(t)\right]dt = \mathscr{L}\{e^{at} f(t)\}$.

❁ EX.5. $\mathscr{L}(e^{at}\cos\omega t) = \dfrac{s-a}{(s-a)^2+\omega^2}$

$$\mathscr{L}(e^{at}\sin\omega t) = \frac{\omega}{(s-a)^2+\omega^2}$$



- Using induction method to prove that the Laplace transform of t^n is $\frac{n!}{s^{n+1}}$.

  By integration by part $[d(uv) = udv + vdu \Rightarrow \int udv = uv - \int vdu]$

$$\mathscr{L}(t^{n+1}) = \int_0^\infty \underbrace{e^{-st}}_{dv}\underbrace{t^{n+1}}_{u}dt = \underbrace{-\frac{1}{s}e^{-st}}_{v}\underbrace{t^{n+1}}_{u}\Big|_0^\infty - \int \frac{-(n+1)}{s}e^{-st}t^n dt$$

$$= 0 + \frac{n+1}{s}\int e^{-st}t^n dt = \frac{n+1}{s}\mathscr{L}(t^n) = \frac{n+1}{s}\frac{n!}{s^{n+1}} = \frac{(n+1)!}{s^{n+2}}.$$

- Prove that $\mathscr{L}(t^a) = \frac{\Gamma(a+1)}{s^{a+1}}$, where gamma function $\Gamma(v) = \int_0^\infty e^{-t}t^{v-1}dt$.

$$\mathscr{L}(t^a) = \int_0^\infty e^{-st}t^a dt \qquad \text{Taking } st = x$$

$$= \int_0^\infty e^{-x}\left(\frac{x}{s}\right)^a \frac{dx}{s}$$

$$= \frac{1}{s^{a+1}}\int_0^\infty e^{-x}x^a dx = \frac{1}{s^{a+1}}\Gamma(a+1).$$

- The Laplace transforms of some important elementary functions are listed in Table 6.1 (on page 224 of the textbook), and a more extensive list is given in Section 6.8 (on pages 264 ~ 267 of the textbook).

- EX.6.

  From Table 6.1 and the first shifting theorem, we immediately obtain another useful formula,

$$\mathscr{L}(t^n e^{at}) = \frac{n!}{(s-a)^{n+1}},$$

and thus

$$\mathscr{L}(t e^{at}) = \frac{1}{(s-a)^2}.$$

Since

(*i*) $\mathscr{L}\{t^n\} = \frac{n!}{s^{n+1}}$

(*ii*) $\mathscr{L}\{e^{at} f(t)\} = F(s-a)$

Existence of Laplace transform

- Piecewise continuous

  A function $f(t)$ is piecewise continuous on a finite interval $a \le t \le b$, if f is continuous on $[a,b]$, except possibly at finitely many points $c_1,\ldots\ldots,c_n$, at each of which f has a finite left and right limit.

- Theorem 3 (Existence theorem for Laplace transforms)

  (i) $f(t)$ is piecewise continuous on $[0, \infty)$

  (ii) There exist some constants k and M such that $|f(t)| \le Me^{kt}$ ← bounded

  $\Rightarrow$ $\mathcal{L}[f](s)$ exists for $s > k$.

Proof

$f(t)$ piecewise continuous on $[0, \infty)$

$\Rightarrow e^{-st}f(t)$ is integrable on $[0, \infty)$.

Assume $s > k$,

$$\Rightarrow \left|\mathscr{L}(f)\right| = \left|\int_0^\infty e^{-st} f(t)dt\right| \le \int_0^\infty |f(t)| e^{-st}dt \le \int_0^\infty Me^{kt}e^{-st}dt$$

$$= \frac{M}{k-s}\left(e^{-(s-k)t}\Big|_0^\infty\right) = \frac{M}{k-s}(0-1) = \frac{M}{s-k}$$

$$\Rightarrow \text{"} s > k \Rightarrow \left|\mathscr{L}(f)\right| < \infty \text{"}.$$

Note 1

Not all function $f(t)$ satisfy $|f(t)| \le Me^{kt}$.

Some functions satisfy the boundary condition; for example,

$\cosh t < e^{t}$, $t^{n} < n!\, e^{t}$ ($n = 0,1,\ldots\ldots$) , …

but we can not find a M and k such that $e^{t^2} < Me^{kt}$

Note 2

The conditions in Theorem 3 are sufficient rather than necessary

(*i.e.*, 充分非必要 or say "satisfy conditions $\underset{\not\Leftarrow}{\Rightarrow}$ $\mathcal{L}[f](s)$ exits" .

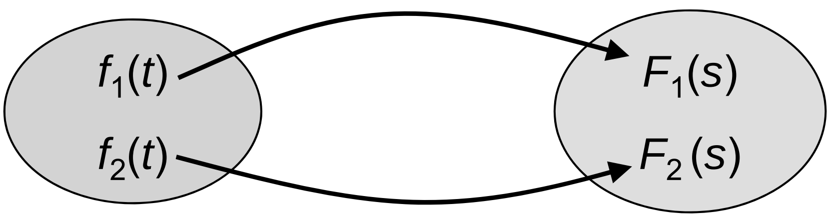
e.g., $\dfrac{1}{\sqrt{t}}$ doesn't satisfy the condition $\left(\dfrac{1}{\sqrt{t}}\Big|_{t=0} = \infty\right)$, but $\mathcal{L}\left[\dfrac{1}{\sqrt{t}}\right]$ exists .

$$\mathcal{L}\left[\frac{1}{\sqrt{t}}\right] = \mathcal{L}\left(t^{-1/2}\right) = \int_0^\infty e^{-st}\, t^{-1/2}\, dt \qquad \text{Taking } st = x$$

$$= \int_0^\infty e^{-x}\left(\frac{x}{s}\right)^{-1/2}\frac{dx}{s} = \left(\frac{1}{s}\right)^{1/2}\int_0^\infty e^{-x}\, x^{-1/2}\, dx = \frac{1}{\sqrt{s}}\,\Gamma\!\left(\frac{1}{2}\right) = \sqrt{\frac{\pi}{s}}\,.$$

Note 3

If the Laplace transform of a given function exists, it is uniquely determined. If $\mathcal{L}[f_1] = \mathcal{L}[f_2]$, then $f_1 = f_2$ on a piecewise interval.



If $F_1(s) = F_2(s)$, then $f_1(t) = f_2(t)$.

It means that Laplace transform is a one-to-one transformation.

- Problems of Section 6.1.

6.2 Transforms of derivatives and integrals

Theorem 1 [$\mathcal{L}(f'(t))$]

1. $f(t)$ continuous on $t \geq 0$,
2. $|f(t)| \leq Me^{kt}$ for some constants k and M
3. $f'(t)$ exists and is piecewise continuous on $t \geq 0$

$\Rightarrow$ $\mathcal{L}(f'(t))$ exits, when $s > k$, and

$\mathcal{L}(f'(t)) = s\,\mathcal{L}(f) - f(0)$.

Proof (integrating by parts)

$$\mathcal{L}(f') = \int_0^\infty e^{-st} f'(t)\,dt = [e^{-st} f(t)]\,|_0^\infty + s\int_0^\infty e^{-st} f(t)\,dt$$

$$= 0 - f(0) + s\,\mathcal{L}(f)$$

$$= s\,\mathcal{L}(f) - f(0).$$

Remark 1.

$$\mathcal{L}(f'') = s\,\mathcal{L}(f') - f'(0)$$

$$= s[s\,\mathcal{L}(f) - f(0)] - f'(0)$$

$$= s^2\mathcal{L}(f) - s\,f(0) - f'(0)$$

$$\mathcal{L}(f''') = s^3\,\mathcal{L}(f) - s^2 f(0) - s\,f'(0) - f''(0)$$

.

.

and *so on*.

Theorem 2 [$\mathcal{L}(f^{(n)}(t))$]

1. $f(t), f'(t), \ldots, f^{(n-1)}(t)$ continuous on $t \geq 0$.
2. $|(f^{(m)}(t)| \leq Me^{kt}$ for $m = 0, 1, 2, \ldots, n-1$.
3. $f^{(n)}(t)$ exists and piecewise continuous on $t \geq 0$.

$\Rightarrow \mathcal{L}((f^{(n)}(t))$ exists, when $s > k$, and

$\mathcal{L}((f^{(n)}(t)) = s^n\,\mathcal{L}(f) - s^{n-1} f(0) - s^{n-2} f'(0) - \ldots - f^{(n-1)}(0)$

❁ Ex.4. $f = t \sin\omega t \Rightarrow \mathcal{L}(t \sin\omega t) = ?$

$f(0) = 0$

$f'(t) = \sin\omega t + \omega t \cos\omega t, \quad f'(0) = 0$

$f''(t) = \omega \cos\omega t + \omega \cos\omega t + \omega^2 t(-\sin\omega t)$

$= 2\omega \cos\omega t - \omega^2 t \sin\omega t$

$= 2\omega \cos\omega t - \omega^2 f(t)$

$\mathcal{L}(f'') = 2\omega \mathcal{L}(\cos\omega t) - \omega^2 \mathcal{L}(f(t))$ (1)

According to Theorem 2, we have

$\mathcal{L}(f'') = s^2 \mathcal{L}(f(t)) - s^2 f(0) - f'(0) = s^2 \mathcal{L}(f(t))$ (2)

From Eqs.(1) and (2), we have

$$\Rightarrow 2\omega \mathcal{L}(\cos\omega t) = (\omega^2 + s^2)\mathcal{L}(f(t))$$

$$\Rightarrow 2\omega \frac{s}{\omega^2 + s^2} = (\omega^2 + s^2)\mathcal{L}(f(t))$$

$$\Rightarrow \mathcal{L}(f(t)) = \mathcal{L}(t \sin\omega t) = \frac{2\omega s}{(\omega^2 + s^2)^2}.$$

Differential equations, initial value problems

❁ Purpose: using Laplace transform to solve *DE*

(initial value problem)

$y'' + ay' + by = r(t), \; y(0) = k_0, \; y'(0) = k_1$

step 1. $\mathcal{L}[y'' + ay' + by] = \mathcal{L}[r(t)]$

⇓ $\mathcal{L}(y) = Y, \; \mathcal{L}(r) = R$

$[s^2 Y - sy(0) - y'(0)] + a[sY - y(0)] + bY = R$

(call the subsidiary equation)

$\Rightarrow (s^2 + as + b)Y(s) = (s + a)\,y(0) + y'(0) + R(s)$

step 2. dividing by ($s^2 + as + b$) and partial fraction

that is, multiply $\dfrac{1}{s^2+as+b} = Q(s).$

$Y(s) = [(s + a)\, y(0) + y'(0)]\, Q(s) + R(s)\, Q(s)$

$= [(s + a)\, k_0 + k_1]\, Q(s) + R(s)\, Q(s)$ ………………… (7)

(partial fraction) reduce Eq.(7) to a sum of terms whose inverses can be found from the table.

step 3. (inverse Laplace transform)

$y(t) = \mathcal{L}^{-1}(Y(s)).$

❁ Ex.5.

$y'' - y = t,\ y(0) = 1,\ y'(0) = 1.$

$$\frac{1}{s^2-1} = \frac{1}{2}\left(\frac{1}{s-1} - \frac{1}{s+1}\right)$$

step 1. (Laplace transform)

$$s^2Y - sy(0) - y'(0) - Y = \frac{1}{s^2}$$

$$\Rightarrow\ (s^2-1)Y = s + 1 + \frac{1}{s^2}.$$

step 2. (dividing by (s^2 - 1) and partial fraction)

$$\Rightarrow Y = \frac{s+1}{s^2-1} + \frac{1}{s^2(s^2-1)}$$

$$= \frac{1}{s-1} + \frac{1}{s^2-1} - \frac{1}{s^2}$$

Hyperbolic sine function

$\sinh x = (e^x - e^{-x})/2.$

Hyperbolic cosine function

$\cosh x = (e^x + e^{-x})/2.$

step 3. (inverse Laplace transform)

$$y(t) = \mathcal{L}^{-1}\left(\frac{1}{s-1}\right) + \mathcal{L}^{-1}\left(\frac{1}{s^2-1}\right) - \mathcal{L}^{-1}\left(\frac{1}{s^2}\right)$$

$$= e^t + \sinh t - t$$

Laplace transform of the integral of a function

Theorem 3 [$\mathcal{L}$ ($\int f(t)dt$)]

1. $f(t)$ is piecewise continuous on $t \geq 0$
2. $|f(t)| \leq Me^{kt}$ for some M and k

$$\Rightarrow \mathcal{L}\left\{\int_0^t f(\tau)d\tau\right\} = \frac{1}{s}\mathcal{L}\{f(t)\} \quad \text{for } s > 0,\ s > k.$$

proof

Let $g(t) = \int_0^t f(\tau)d\tau$, $g(t)$ is continuous.

$$|g(t)| \leq \int_0^t |f(\tau)|d\tau \leq M\int_0^t e^{k\tau}d\tau = \frac{M}{k}(e^{kt} - 1) \leq \frac{M}{k}e^{kt} \quad (k > 0)$$

That is $g(t)$ also satisfies the condition $|g(t)| \leq N e^{kt}$ for some N and k. Thus $\mathcal{L}\{g(t)\}$ exists. $g'(t) = f(t)$, except for points at which $f(t)$ is discontinuous. Hence $g'(t)$ is piecewise continuous. By theorem 1, $\mathcal{L}\{f(t)\} = \mathcal{L}\{g'(t)\} = s\,\mathcal{L}\{g(t)\} - g(0)$, clearly $g(0) = 0$. so that $\mathcal{L}(f) = s\,\mathcal{L}\{g(t)\}$

$$\Rightarrow \mathcal{L}\left\{\int_0^t f(\tau)d\tau\right\} = \frac{1}{s}\mathcal{L}\{f(t)\}.$$

- Remark

$$\mathcal{L}\left\{\int f(\tau)d\tau\right\} = \frac{1}{s}\mathcal{L}\{f\}$$

$$\Rightarrow \mathcal{L}^{-1}\left\{\frac{1}{s}\mathcal{L}\{f\}\right\} = \int f(\tau)d\tau.$$

- Ex.7. $\mathcal{L}(f) = \dfrac{1}{s(s^2+\omega^2)} \Rightarrow f(t) = ?$

By Table 6.1 $\mathcal{L}^{-1}\left(\dfrac{1}{s^2+\omega^2}\right) = \dfrac{1}{\omega}\sin\omega t$

$$\mathcal{L}^{-1}\left\{\frac{1}{s}\frac{1}{s^2+\omega^2}\right\} = \mathcal{L}^{-1}\left\{\frac{1}{s}\mathcal{L}\left(\frac{1}{\omega}\sin\omega t\right)\right\} = \int_0^t \frac{1}{\omega}\sin\omega\tau\, d\tau$$

$$= \frac{1}{\omega^2}(-\cos\omega\tau)\Big|_{\tau=0}^{t} = \frac{1}{\omega^2}(1-\cos\omega t).$$

- Ex.8. $\mathcal{L}(f)=\dfrac{1}{s^2(s^2+\omega^2)} \Rightarrow f(t)=?$

$$\mathcal{L}^{-1}\left(\frac{1}{s^2}\frac{1}{s^2+\omega^2}\right)=\mathcal{L}^{-1}\left\{\frac{1}{\omega^2}(\frac{1}{s^2}-\frac{1}{s^2+\omega^2})\right\}$$
$$=\frac{1}{\omega^2}\left\{\mathcal{L}^{-1}(\frac{1}{s^2})-\mathcal{L}^{-1}(\frac{1}{s^2+\omega^2})\right\}$$
$$=\frac{1}{\omega^2}(t-\frac{1}{\omega}\sin\omega t)$$

or

$$\mathcal{L}^{-1}\left(\frac{1}{s^2}(\frac{1}{s^2+\omega^2})\right)=\mathcal{L}^{-1}\left\{\frac{1}{s}[\frac{1}{s}\frac{1}{s^2+\omega^2}]\right\}$$
$$=\int_0^t\frac{1}{\omega^2}(1-\cos\omega\tau)d\tau$$
$$=\frac{1}{\omega^2}(\tau-\frac{1}{\omega}\sin\omega\tau)\Big|_{\tau=0}^{t}$$
$$=\frac{1}{\omega^2}(t-\frac{1}{\omega}\sin\omega t).$$

$$\mathcal{L}(f)=\frac{1}{s}\left(\frac{1}{s}\frac{1}{s^2+\omega^2}\right)$$
$$f=\int_0^t f_1(\tau)d\tau$$
where $\mathcal{L}(f_1)=\dfrac{1}{s(s^2+\omega^2)}$
$$\mathcal{L}(f_1)=\frac{1}{s}\frac{1}{s^2+\omega^2}$$
$$f_1=\int_0^t f_2(\tau)d\tau$$
where $\mathcal{L}(f_2)=\dfrac{1}{s^2+\omega^2}$
$$f=\iint f_2$$
where $f_2=\dfrac{1}{\omega}\sin\omega t$

- Shifted data problem

  = an initial value problem with initial conditions refer to some later constant instead of $t = 0$.

  For example, $y'' + ay' + by = r(t)$, $y(t_1) = k_1$, $y'(t_1) = k_2$.

- Ex.9. $y''+y=2t,\quad y(\frac{1}{4}\pi)=\frac{1}{2}\pi,\quad y'(\frac{1}{4}\pi)=2-\sqrt{2}.$

step 1. $s^2Y-sy(0)-y'(0)+Y=2\dfrac{1}{s^2}$

$$(s^2+1)Y=sy(0)+y'(0)+\frac{2}{s^2}.$$

step 2. $Y=\dfrac{2}{s^2(s^2+1)}+y(0)\dfrac{s}{s^2+1}+y'(0)\dfrac{1}{s^2+1}$

$$=\frac{2}{s^2}-\frac{2}{s^2+1}+y(0)\frac{s}{s^2+1}+y'(0)\frac{1}{s^2+1}.$$

step 3. $y = \mathcal{L}^{-1}(Y) = 2t - 2\sin t + y(0)\cos t + y'(0)\sin t$

$= 2t + y(0)\cos t + (y'(0) - 2)\sin t$

Let $A = y(0)$, $B = y'(0) - 2$

$= 2t + A\cos t + B\sin t$

since $y(\frac{1}{4}\pi) = \frac{1}{2}\pi \Rightarrow \frac{1}{2}\pi + A\frac{1}{\sqrt{2}} + B\frac{1}{\sqrt{2}} = \frac{1}{2}\pi$

$y'(\frac{1}{4}\pi) = 2 - \sqrt{2} \Rightarrow (2 - A\sin t + B\cos t)|_{t=\frac{1}{4}\pi} = 2 - \sqrt{2}$

$\Rightarrow 2 - A\frac{1}{\sqrt{2}} + B\frac{1}{\sqrt{2}} = 2 - \sqrt{2}$

- Problems of Section 6.2.

$$\Rightarrow \begin{cases} A\frac{1}{\sqrt{2}} + B\frac{1}{\sqrt{2}} = 0 \\ A\frac{1}{\sqrt{2}} - B\frac{1}{\sqrt{2}} = \sqrt{2} \end{cases} \Rightarrow \begin{cases} A + B = 0 \\ A - B = 2 \end{cases} \Rightarrow \begin{cases} A = 1 \\ B = -1 \end{cases}$$

$\Rightarrow y = 2t + \cos(t) - \sin t.$

- The Hermann grid illusion