5. Series Solutions of *ODEs*

5.1 Power series method and Theory of the power series method

5.1 Power series method

- The power series method is the standard basic method for solving linear differential equations with variable coefficients.
- Power series

$$\sum_{m=0}^{\infty} a_m (x - x_0)^m = a_0 + a_1(x - x_0) + a_2(x - x_0)^2 + \dots ,$$

where a_i's are constants, called the coefficients of the series, x_0 is a constant, called the center of series, and x is a variable.

- For example,

$$\frac{1}{1-x} = \sum_{m=0}^{\infty} x^m \qquad (|x| < 1)$$

$$e^x = \sum_{m=0}^{\infty} \frac{x^m}{m!}$$

Taylor series

$$f(x) = \sum_{n=0}^{\infty} \frac{f^{(n)}(x_0)}{n!}(x - x_0)^n$$

$$\cos x = \sum_{m=0}^{\infty} \frac{(-1)^m x^{2m}}{(2m)!}$$

$$\sin x = \sum_{m=0}^{\infty} \frac{(-1)^m x^{2m+1}}{(2m+1)!}.$$

Idea of the power series method

For a given equation

$y'' + p(x)\, y' + q(x)\, y = 0$

step 1. Assume solution $y(x)$ in the form of a power series with unknown coefficients

$$y(x) = \sum_{m=0}^{\infty} a_m x^m \quad \text{or} \quad \sum_{m=0}^{\infty} a_m (x - x_0)^m$$

step 2..Represent $p(x)$ and $q(x)$ by power series in powers of x (or of $(x\text{-}x_0)$ if solutions in powers of $(x\text{-}x_0)$ are wanted).

step 3. Substituting $y(x)$ into the original *DE* to find out all coefficients a_m's.

Ex. 2.

$$y' - y = 0$$

$$y = \sum_{m=0}^{\infty} a_m x^m$$

$$\Rightarrow (a_1 + 2a_2 x + 3a_3 x^2 + \ldots) - (a_0 + a_1 x + a_2 x^2 + \ldots) = 0$$

$$\Rightarrow (a_1 - a_0) + (2a_2 - a_1)x + (3a_3 - a_2)x^2 + \ldots = 0$$

$$\Rightarrow a_1 - a_0 = 0, \quad 2a_2 - a_1 = 0, \quad 3a_3 - a_2 = 0$$

$$\Rightarrow a_1 = a_0$$

$$a_2 = \frac{1}{2}a_1 = \frac{1}{2}a_0 = \frac{1}{2!}a_0$$

$$a_3 = \frac{1}{3}a_2 = \frac{1}{6}a_0 = \frac{1}{3!}a_0$$

$$\vdots$$

$$\Rightarrow y = a_0 + a_0 x + \frac{a_0}{2!}x^2 + \frac{a_0}{3!}x^3 + \ldots$$

$$= a_0\left(1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \ldots\right)$$

$$= a_0\, e^x.$$

Theory of the power series method

- Basic concepts

Power series is an infinite series of the form

$$\sum_{m=0}^{\infty} a_m (x - x_0)^m = a_0 + a_1(x - x_0) + a_2(x - x_0)^2 + \dots\dots\dots\dots \quad (1)$$

The *nth* partial sum of Eq.(1) is

$$S_n(x) = a_0 + a_1(x - x_0) + \dots + a_2(x - x_0)^n,$$

where n = 0, 1, 2, …..

Let $R_n(x) = a_{n+1}(x\text{-}x_0)^{n+1} + a_{n+2}(x\text{-}x_0)^{n+2} + \dots$

$R_n(x)$ is the remainder of Eq.(1) after the term $a_n(x\text{-}x_0)^n$.

That is $\sum_{m=0}^{\infty} a_m (x - x_0)^m = S_n(x) + R_n(x), \quad n = 0, 1, 2, \dots$

- Consider the sequence of the partial sums $S_0(x)$, $S_1(x)$, $S_2(x)$, …

If for some $x = x_1$, this sequence converges, say, $\lim_{n \to \infty} S_n(x_1) = S(x_1)$,

then the series (1) is convergent at $x = x_1$, the number $S(x_1)$ is called the value or sum of Eq.(1) at x_1, and written

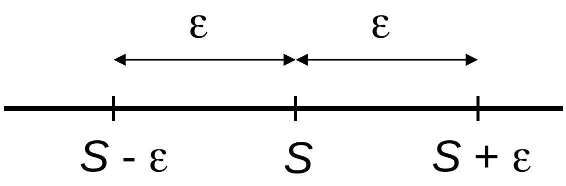
$$S(x_1) = \sum_{m=0}^{\infty} a_m (x_1 - x_0)^m$$

Then we have for every n ,

$$S(x_1) = S_n(x_1) + R_n(x_1) \quad \dots\dots\dots\dots\dots\dots\dots\dots \quad (4)$$

If that sequence diverges at $x = x_1$, the series (1) is called divergent at $x = x_1$.

- In the case of convergence, for any positive ε, there is an N (depending on ε) such that, by Eq.(4), $|R_n(x_1)| = |S(x_1) - S_n(x_1)| < \varepsilon$ for all $n > N$



- Convergence interval and radius of convergence
  Three cases of convergence

  consider the series $S(x) = \sum_{m=0}^{\infty} a_m (x - x_0)^m$

  case 1: The series converges only at $x = x_0$.

  case 2: The series converges when $| x - x_0 | < R$, and diverges when $|x - x_0| \geq R$. The interval $|x - x_0|$ is called the convergence interval, and R is called the radius of convergence of the series, R can be obtained from either of the formulas

$$(a)\ R = \frac{1}{\lim_{m\to\infty} \sqrt[m]{|a_m|}}$$

or

$$(b)\ R = \frac{1}{\lim_{m\to\infty} \left|\frac{a_{m+1}}{a_m}\right|}$$

provided these limits exits and are not zero.

case 3: The convergence interval is infinite; that is, the series converges for all x. In other words , $R = \infty$.

- Ex. $\sum_{m=0}^{\infty} m!\, x^m$, $R = \frac{1}{\lim_{m\to\infty} \left|\frac{(m+1)!}{m!}\right|} = \frac{1}{\lim_{m\to\infty} (m+1)} = 0.$

  That is the series converges only at $x = 0$.

- Ex. $\sum_{m=0}^{\infty} x^m$, $R = \frac{1}{\lim_{m\to\infty} \left|\frac{1}{1}\right|} = 1.$

  The series converges when $|x| < 1$.

- Ex. $\sum_{m=0}^{\infty} \frac{x^m}{m!}$, $a_m = \frac{1}{m!}$, $R = \frac{1}{\lim_{m\to\infty}\left|\frac{a_{m+1}}{a_m}\right|} = \frac{1}{\lim_{m\to\infty} \frac{1}{m+1}} = \infty$

  The series converges for all *x.*

- Ex. $\sum_{m=0}^{\infty} \frac{(-1)^m}{8^m} x^{3m}$ Let $t = x^3$, then consider the convergence interval of $\sum_{m=0}^{\infty} \frac{(-1)^m}{8^m} t^m$.

$$R = \frac{1}{\lim_{m\to\infty}\left|\frac{a_{m+1}}{a_m}\right|} = \frac{1}{\lim_{m\to\infty}\left|\frac{(-1)^{m+1}/8^{m+1}}{(-1)^m/8^m}\right|} = \frac{1}{\lim_{m\to\infty}\left|\frac{-1}{8}\right|} = 8$$

The series $\sum \frac{(-1)^m}{8^m} t^m$ converges for $|t| < 8$; that is,

the series $\sum \frac{(-1)^m}{8^m} x^{3m}$ converges for $|x^3| < 8 \equiv |x| < 2$.

Operations on power series

- Termwise differentiation

  That is a power series may be differentiated term by term.

  If $y(x) = \sum_{m=1}^{\infty} a_m (x - x_0)^m$

  converges for $|x - x_0| < R$, where $R > 0$, then the series obtained by differentiating term by term also converges for those *x* and represents the derivative *y*' of *y* for those *x*; that is

$$y'(x) = \sum_{m=1}^{\infty} m a_m (x - x_0)^{m-1}, \quad |x - x_0| < R,$$

$$y''(x) = \sum_{m=2}^{\infty} m(m-1) a_m (x - x_0)^{m-2}, \quad |x - x_0| < R,$$

  and so on.

- Termwise addition

Two power series may be added term by term. That is at points common to both intervals of convergence about x_0 , two power series can be added or subtracted term by term , and the resulting series converges in the common interval of convergence.

$$\sum_{m=0}^{\infty} a_m (x-x_0)^m + \sum_{m=0}^{\infty} b_m (x-x_0)^m = \sum_{m=0}^{\infty} (a_m + b_m)(x-x_0)^m .$$

- Termwise Multiplication

Two power series may be multipled term by term.

$$\sum_{m=0}^{\infty} a_m (x-x_0)^m \sum_{m=0}^{\infty} b_m (x-x_0)^m = \sum_{n=0}^{\infty} \sum_{k=0}^{\infty} a_k b_{n-k} (x-x_0)^m$$

$$= a_0 b_0 + (a_0 b_1 + a_1 b_0)(x-x_0) + (a_0 b_2 + a_1 b_1 + a_2 b_0)(x-x_0)^2 + \ldots$$

The resulting series converges at points common to the interior of the intervals of convergence of both *f* and *g.*

- Termwise multiplication of a power series by a constant

$$k \sum_{m=0}^{\infty} a_m \left(x - x_0\right)^m = \sum_{m=0}^{\infty} k\, a_m \left(x - x_0\right)^m$$

has the same convergence interval as $\sum_{m=0}^{\infty} a_m \left(x - x_0\right)^m$.

- Vanishing of all coefficients

$$\sum_{m=0}^{\infty} a_m (x-x_0)^m \text{ convergeswhen } | x - x_0 | < R$$

and $\sum_{m=0}^{\infty} a_m (x-x_0)^m = 0 \quad$ for all $| x - x_0 | < R$

$\Rightarrow$ $a_m = 0$ for all m .

Shifting summation indices

$$\sum_{m=2}^{\infty} m(m-1)\,a_m x^{m-2}$$

$$\Downarrow \quad \text{set} \quad s = m - 2 \quad \Rightarrow \quad m = s + 2$$

$$\sum_{s=0}^{\infty} (s+2)(s+1)\,a_{s+2}\,x^s$$

This is needed, for instance, in writing the sum of two series as a single series.

Example

$$x^2 \sum_{m=2}^{\infty} m(m-1)\,a_m x^{m-2} + 2\sum_{m=1}^{\infty} m\,a_m\,x^{m-1}$$

$$= \sum_{m=2}^{\infty} m(m-1)\,a_m\,x^m + 2\sum_{m=1}^{\infty} m\,a_m\,x^{m-1}$$

$$\Downarrow \; s = m \qquad \Downarrow \; s = m - 1 \quad \Rightarrow \quad m = s + 1$$

$$= \sum_{s=2}^{\infty} s(s-1)\,a_s\,x^s + \sum_{s=0}^{\infty} 2(s+1)\,a_{s+1}\,x^s$$

$$= \sum_{s=0}^{\infty} \left[s(s-1)\,a_s + 2(s+1)\,a_{s+1} \right] x^s \,.$$

- Existence of power series solutions

  Question: Whether an equation has power series solutions al all ?

  Answer: If the coefficients p, q and r of $y'' + p(x)y' + q(x)y = r(x)$ have power series representations, then the equation has power series solutions.

- Definition of real analytic functions

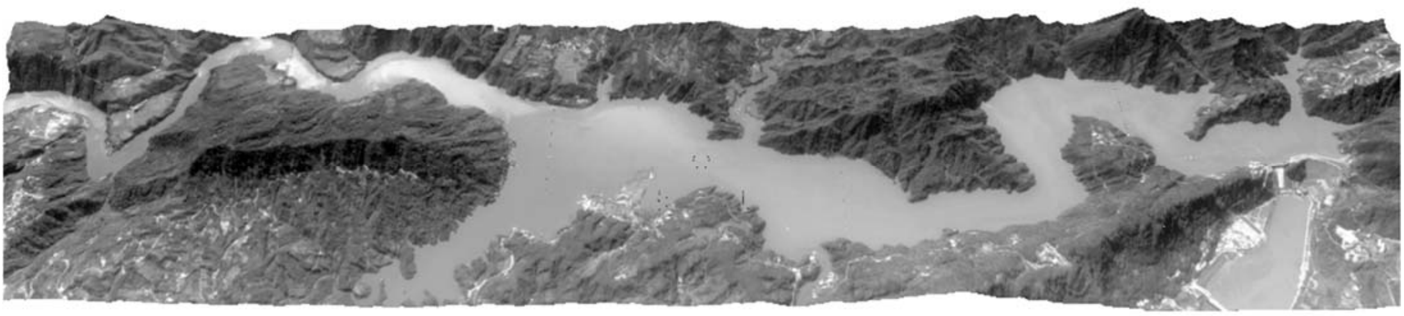
  A real function $f(x)$ is called analytic at a point $x = x_0$, if it can be represented by a power series in powers of $x - x_0$ with radius of convergence $R > 0$.

  With this we have

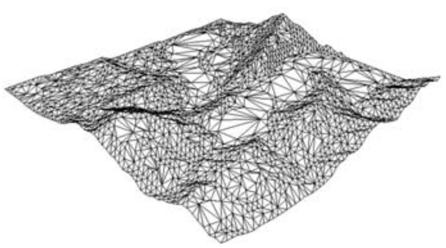
  Theorem 1 (Existence of power series solution)
  If p, q and r of equation $y'' + p(x)y' + q(x)y = r(x)$ are analytic at $x = x_0$, then every solution of the equation is analytic at $x = x_0$ and can thus be represented by a power series in powers of $x - x_0$ with radius of convergence $R > 0$.

- Problems of Section 5.1.

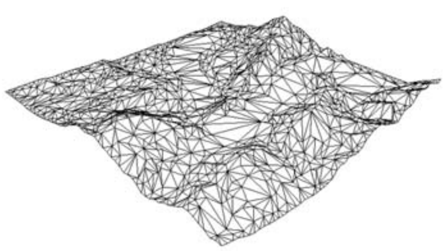
Multiresolution terrain modeling



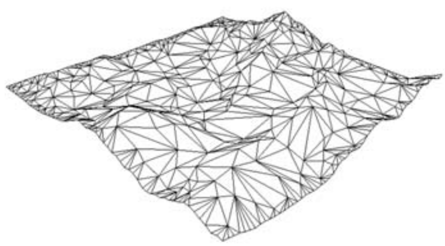
Shyr-Men dam of area 2.5*km*×12.5*km* mapped with an airborne image.



3,870 triangles.



1,932 triangles.



782 triangles.