3. Higher Order Linear *ODEs*

3.1 Homogeneous linear *ODEs*

- Definition of high-order linear *DE*

$$y^{(n)} + p_{n-1}(x)\, y^{(n-1)} + \ldots\ldots + p_1(x)\, y' + p_0(x)\, y = r(x)$$

is a non-homogeneous linear equation

$$y^{(n)} + p_{n-1}(x)\, y^{(n-1)} + \ldots\ldots + p_1(x)\, y' + p_0(x)\, y = 0$$

is the corresponding homogeneous linear equation.

- Solutions of the homogeneous equation

(*i*) general solution $y(x) = c_1 y_1(x) + c_2 y_2(x) + \ldots + c_n y_n(x)$
where c_i's are arbitrary constant, y_i's is a basis of solutions.

(*ii*) particular solution, if c_i's are specific.

- Theorem 1 (Superposition principle or linearity principle)
Any linear combination of homogeneous solutions is again a homogeneous solution.

- Warning: Theorem 1 does not hold for non-homogeneous equation or a nonlinear equation.

- Linear independent

  $y_1(x), y_2(x), \ldots, y_n(x)$ are linear independent

  $\Leftrightarrow c_1y_1 + c_2y_2 + \ldots\ldots + c_ny_n = 0 \Rightarrow c_1 = c_2 = c_3 = \ldots = c_n = 0.$

  Otherwise, y_i's are linear dependent.

- Initial value problem

$$y^{(n)} + p_{n-1}(x)\, y^{(n-1)} + \ldots\ldots + p_1(x)\, y' + p_0(x)\, y = 0$$

$$\left.\begin{aligned} y(x_0) &= k_0 \\ y'(x_0) &= k_1 \\ &\vdots \\ y^{(n-1)}(x_0) &= k_{n-1} \end{aligned}\right\} \text{ initial conditions}$$

- Theorem 2 (Existence and uniqueness theorem)

  If $p_0(x), \ldots, p_{n-1}(x)$ are continuous functions on some open interval I and x_0 in I, then the initial value problem has a unique solution $y(x)$ on the interval I.

- Linear independence of solutions

  Wronskian $W(y_1, y_2, \ldots, y_n) = \begin{vmatrix} y_1 & y_2 & \cdots & y_n \\ y_1' & y_2' & & y_n' \\ \vdots & \vdots & & \vdots \\ y_1^{(n-1)} & y_2^{(n-1)} & \cdots & y_n^{(n-1)} \end{vmatrix}$

- Theorem 3

  If $p_o(x), \ldots, p_{n-1}(x)$ are continuous, then n solutions $y_1(x), \ldots, y_n(x)$,

  (*a*) y_i's are linear dependent $\Leftrightarrow W[y_i\text{'s}] = 0.$

  (*b*) if $W[x_0] = 0$, x_0 in $I \Rightarrow W[x] \equiv 0$ for all x in I

  if $W[x_0] \neq 0$, x_0 in $I \Rightarrow W[x] \neq 0$ for all x in I.

  Proof: The same as the proof of Theorem 2 of Section 2.6.

- A general solution of homogeneous equation

- Theorem 4 and 5 (Existence of a general solution)

  If $p_0(x), p_1(x), \ldots, p_{n-1}(x)$ are continuous on some open interval I, then the homogeneous equation has a general solution of the form

$$y(x) = c_1y_1 + c_2y_2 + \ldots + c_ny_n \quad \text{on } I_0.$$

- Problems of Section 3.1.

3.2 Homogeneous linear *ODEs* with constant coefficients

- For a homogeneous equation

$$y^{(n)} + a_{n-1}y^{(n-1)} + \ldots + a_1y' + a_0y = 0 \quad \ldots\ldots\ldots\ldots\ldots\ldots \quad (1)$$

substituting $y(x) = e^{\lambda x}$ into Eq.(1) to obtain the characteristic equation

$$\lambda^n + a_{n-1}\lambda^{n-1} + \ldots + a_1\lambda + a_0 = 0 \quad \ldots\ldots\ldots\ldots\ldots\ldots \quad (2)$$

- Consider the roots of the characteristic equation

- *Case* 1. Distinct real roots

  If Eq.(2) has n real unequal roots $\lambda_1, \lambda_2, \lambda_3, .., \lambda_n$, then the n solutions $y_1 = e^{\lambda 1x}$, $y_2 = e^{\lambda 2x}$, … , $y_n = e^{\lambda nx}$ constitute a basis for all x, and the corresponding general solution of Eq.(1) is

$$y = c_1y_1 + c_2y_2 + \ldots + c_ny_n.$$

- Theorem 1. If all λ_i's are different, then $e^{\lambda_i x}$'s are linearly independent.

Proof.

$$W = \begin{vmatrix} e^{\lambda_1 x} & e^{\lambda_2 x} & \cdots & e^{\lambda_n x} \\ \lambda_1 e^{\lambda_1 x} & \lambda_2 e^{\lambda_2 x} & \cdots & \lambda_n e^{\lambda_n x} \\ \vdots & \cdots & \cdots & \vdots \\ \lambda_1^{n-1} e^{\lambda_1 x} & \cdots & \cdots & \lambda_n^{n-1} e^{\lambda_n x} \end{vmatrix}$$

$$= e^{(\lambda_1+\lambda_2+\ldots+\lambda_n)x} \begin{vmatrix} 1 & 1 & \cdots & 1 \\ \lambda_1 & \lambda_2 & \cdots & \lambda_n \\ \vdots & \cdots & \cdots & \vdots \\ \lambda_1^{n-1} & \cdots & \cdots & \lambda_n^{n-1} \end{vmatrix}$$

The determinant is called Vandermonde or Cauchy determinant

$$= e^{(\lambda_1+\ldots+\lambda_n)x}(-1)^{\frac{n(n-1)}{2}}(\lambda_1-\lambda_2)\,(\lambda_1-\lambda_3)\cdots(\lambda_1-\lambda_n)\,(\lambda_2-\lambda_3)\cdots(\lambda_2-\lambda_n)\cdots(\lambda_{n-1}-\lambda_n)$$

$$\neq 0.$$

- *Case* 2. Simple complex roots (有一對共軛複數根)
If complex roots occur, they must occur in conjugate pairs since the coefficients of Eq.(1) are real. If $\lambda = v + i\,w$ is a simple root of Eq.(2) so is the conjugate $\lambda = v - i\,w$, and two corresponding linearly independent solution are $y_1 = e^{vx}\cos wx$, $y_2 = e^{vx}\sin wx$.

- Ex.2.

$$y''' - 2y'' + 2y' = 0.$$

$$\Rightarrow \lambda^3 - 2\lambda^2 + 2\lambda = 0.$$

$$\Rightarrow \lambda_1 = 0,\ \lambda_2 = 1 + i,\ \lambda_3 = 1 - i.$$

$$\Rightarrow y_1 = 1,\ y_2 = e^{x}\cos x,\ y_3 = e^{x}\sin x.$$

- *Case* 3. Multiple real roots

➢ If a real double root occurs, say $\lambda_1 = \lambda_2$, then taking $y_1 = e^{\lambda_1 x}$ and $y_2 = x e^{\lambda_1 x}$.

➢ If a triple root occurs, say $\lambda_1 = \lambda_2 = \lambda_3$, then taking $y_1 = e^{\lambda_1 x}$, $y_2 = x e^{\lambda_1 x}$, $y_3 = x^2 e^{\lambda_1 x}$.

➢ If λ_1 is a root of order m, then m corresponding linearly independent solutions are $e^{\lambda_1 x}, x e^{\lambda_1 x}, \ldots , x^{m-1} e^{\lambda_1 x}$.

- How to obtain the solutions $e^{\lambda_1 x}, x e^{\lambda_1 x}, \ldots , x^{m-1} e^{\lambda_1 x}$?

$y^{(n)} + a_{n-1} y^{(n-1)} + \ldots + a_1 y' + a_0 y = 0$

$\Rightarrow [D^n + a_{n-1} D^{n-1} + \ldots + a_0] \, y = 0$

$\Rightarrow L[y] = 0$

$\Rightarrow L[e^{\lambda x}] = (\lambda^n + a_{n-1} \lambda^{n-1} + \ldots + a_0) \, e^{\lambda x}$

Let λ_1 be an *m th* order root of the polynomial, and let $\lambda_{m+1}, \ldots , \lambda_n$ be other roots, all different to λ_1, when $m < n$.

$\Rightarrow L[e^{\lambda x}] = (\lambda - \lambda_1)^m h(\lambda) \, e^{\lambda x}$.

Differentiating the equation *w.r.t.* λ

$$\Rightarrow \frac{\partial}{\partial \lambda} L[e^{\lambda x}] = \frac{\partial}{\partial \lambda}[(\lambda - \lambda_1)^m h(\lambda) e^{\lambda x}]$$

$$\Rightarrow L[\frac{\partial}{\partial \lambda} e^{\lambda x}] = m(\lambda - \lambda_1)^{m-1} h(\lambda) e^{\lambda x} + (\lambda - \lambda_1)^m \frac{\partial}{\partial \lambda}[h(\lambda) e^{\lambda x}]$$

$$\Rightarrow L[x e^{\lambda x}] = m(\lambda - \lambda_1)^{m-1} h(\lambda) e^{\lambda x} + (\lambda - \lambda_1)^m \frac{\partial}{\partial \lambda}[h(\lambda) e^{\lambda x}] \quad \Leftarrow$$

$$\Rightarrow L[x e^{\lambda_1 x}] = 0 \quad \text{if} \quad m \geq 2$$

$$\Rightarrow x e^{\lambda_1 x} \text{ is a solution of Eq.(1).}$$

We can repeat this step and produce $x^2 e^{\lambda_1 x}, \ldots , x^{m-1} e^{\lambda_1 x}$, by another $(m-2)$ such differentiation with respect to λ

$$\frac{\partial^m}{\partial \lambda^m} L[e^{\lambda_1 x}] = m!\,(\lambda_1 - \lambda_1)^0 h(\lambda_1) e^{\lambda_1 x} + \ldots + (\lambda - \lambda_1)^k \neq 0,\ k \geq 1,$$

$$\Rightarrow L[x^m e^{\lambda_1 x}] \neq 0$$

There is no longer produce solution, so we get precisely the solutions $e^{\lambda 1x}$, $x\, e^{\lambda 1x}$, … , $x^{m-1} e^{\lambda 1x}$.

- *Case* 4. Multiple complex roots (多對相同共軛樹根)

  If $\lambda = v + iw$ is a complex double root, so is the conjugate $\bar{\lambda} = v - iw$. The corresponding linearly independent solution are $e^{vx}\cos wx$, $e^{vx}\sin wx$, $xe^{vx}\cos wx$, $xe^{vx}\sin wx$.

  If λ is a complex triple root, the corresponding linearly independent solutions are

  $e^{vx}\cos wx$, $e^{vx}\sin wx$, $xe^{vx}\cos wx$, $xe^{vx}\sin wx$,

  $x^2e^{vx}\cos wx$, $x^2e^{vx}\sin wx$.

- Problems of Section 3.2.

3.3 Nonhomogeneous linear *ODEs*

$$y^{(n)} + p_{n-1}\, y^{(n-1)} + \ldots + p_1(x)\, y' + p_0(x)\, y = r(x) \quad \ldots\ldots\ldots\ldots\ldots\ldots \quad (1)$$

The corresponding homogeneous equation

$$y^{(n)} + p_{n-1}\, y^{(n-1)} + \ldots + p_1(x)\, y' + p_0(x)\, y = 0 \quad \ldots\ldots\ldots\ldots\ldots\ldots\ldots \quad (2)$$

- Theorem 1. Let y_1 and y_2 be arbitrary solution of Eq.(1) , and y_h is an arbitrary solution of Eq.(2)

  (*a*) $(y_1 - y_2)$ is a solution of Eq.(2)

  (*b*) $(y_1 + y_h)$ is a solution of Eq.(1).

- A general solution of the non-homogeneous equation (1) on some open interval I is a solution of the form

  $$y(x) = y_h(x) + y_p(x) \quad \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots \quad (3)$$

  where $y_h(x) = c_1\, y_1(x) + \ldots + c_n\, y_n(x)$ is a general solution of the homogeneous equation (2).

- A particular solution of Eq.(1) on I is a solution obtained from Eq.(3) by assigning specific values to the arbitrary constants $c_1, c_2, \dots, c_n$ in $y_h(x)$

- Theorem 2 (General solution)
  If all coefficients $p_i(x)$'s , $0 \le i \le n-1$, and $r(x)$ are continuous on some open interval I, then Eq.(1) has a general solution on I.

- Initial value problem

$$y^{(n)} + p_{n-1}(x)\, y^{(n-1)} + \dots + p_0\, y = r(x)$$

$$\left.\begin{aligned} y(x_0) &= k_0 \\ y'(x_0) &= k_1 \\ &\ \vdots \\ y^{(n-1)}(x_0) &= k_{n-1}. \end{aligned}\right\} \quad \text{initial conditions}$$

- Theorem 3 If $p_i(x)$'s and $r(x)$ are continuous on some open interval I. Then Eq.(4) has a unique solution on I_0.

Method of undetermined coefficients

$$y^{(n)} + p_{n-1}y^{(n-1)} + \dots + p_0(x)\, y = r(x)$$

- General solution is $y = y_h(x) + y_p(x)$,
  where $y_h(x)$ is the general solution of the corresponding homogeneous equation. $y_p(x)$ is the particular solution of the nonhomogeneous equation. Find $y_p(x)$ by choosing a form similar to that of $r(x)$ and determining the associated unknown coefficients.

- How to choose the form of $y_p(x)$?
  (*a*) Basis rule (as in Section 2.7)
  (*b*) Modification rule
  if $cr(x)$ = a solution of the corresponding homogeneous equation (2), then multiply $y_p(x)$ by x^k, where k is the smallest positive integer such that no term of $x^k y_p(x)$ is a solution of Eq.(2).
  (*c*) Sum rule
  If $r(x) \equiv$ linear combination of functions listed in the left column of listed Table 2.1 of Section 2.7, then $y_p(x)$ = linear combination of the corresponding functions in the right column of Table 2.1.

- Solving initial value problem
  step 1. solving $y_h(x)$
  step 2. solving $y_p(x)$
  step 3. taking $y_h(x) + y_p(x)$ to satisfy the initial conditions.

Method of variation of parameters

$$y^{(n)} + p_{n-1}(x)\, y^{(n-1)} + \dots + p_0(x)\, y = r(x) \qquad \text{(1)}$$

$$y^{(n)} + p_{n-1}(x)\, y^{(n-1)} + \dots + p_0(x)\, y = 0 \qquad \text{(2)}$$

$$\begin{aligned} y_p(x) = {} & y_1(x) \int \frac{W_1(x)}{W(x)} r(x)\, dx \\ & + y_2(x) \int \frac{W_2(x)}{W(x)} r(x)\, dx \\ & + \dots \\ & + y_n(x) \int \frac{W_n(x)}{W(x)} r(x)\, dx \end{aligned}$$

Here , $y_1, y_2, \dots , y_n$ is a basis of solution of the corresponding homogeneous equation on I , W is the Wronskian of y_i*'s,* and W_j ($j = 1, \dots , n$) is obtained from W by replacing the j th column of W by the column $[0 \dots 0\ 1]^T$.

$$W = \begin{vmatrix} y_1 & y_2 & \dots & y_n \\ y_1' & \dots & \dots & y_n' \\ \dots & \dots & \dots & \dots \\ y_1^{(n-1)} & \dots & \dots & y_n^{(n-1)} \end{vmatrix} \Rightarrow W_j = \begin{vmatrix} y_1 & \dots & y_{j-1} & 0 & y_{j+1} & \dots & y_n \\ y_1' & \dots & y'_{j-1} & 0 & y'_{j+1} & \dots & y'_n \\ \dots & \dots & \dots & \dots & \dots & \dots & \dots \\ y_1^{(n-1)} & \dots & y_{j-1}^{(n-1)} & 1 & y_{j+1}^{(n-1)} & \dots & y_n^{(n-1)} \end{vmatrix}$$

- The proof is the same as the idea in Section 2.10.

$$y^{(n)} + p_{n-1}(x)\, y^{(n-1)} + \dots + p_1(x)\, y' + p_0(x)\, y = 0.$$

$\equiv L[y] = 0$, $y = c_1 y_1 + \dots + c_n y_n$.

Taking $y_p = u_1(x)\, y_1 + \dots + u_n(x)\, y_n$, then to solve $u_i(x)$'s.

We need n conditions to solve $u_i(x)$'s. Now we just have one condition

$$y_p^{(n)} + p_{n-1}\, y_p^{(n-1)} + \dots + p_0 y_p = r(x) ,$$

To derive other n - 1 conditions,

$$y_p' = (u_1'y_1 + \dots + u_n'y_n) + (u_1 y_1' + \dots + u_n y_n')$$

Take $u_1'y_1 + \dots + u_n'y_n = 0$ as the 1st condition,

$$y_p'' = (u_1'y_1' + \dots + u_n'y_n') + (u_1 y_1'' + \dots + u_n y_n'')$$

Take $u_1'y_1' + \dots + u_n'y_n' = 0$ as the 2nd condition.

…

and so on, until

$$y_p^{(n-1)} = (u_1'y_1^{(n-2)} + \dots + u_n'y_n^{(n-2)}) + (u_1 y_1^{(n-1)} + \dots + u_n y_n^{(n-1)})$$

Take $u_1'y_1^{(n-2)} + \dots + u_n'y_n^{(n-2)} = 0$ as the (n-1) th condition.

Differentiating $y_p^{(n-1)}(x)$ to get

$$y_p^{(n)} = (u_1 y_1^{(n)} + \dots + u_n y_n^{(n)}) + (u_1'y_1^{(n-1)} + \dots + u_n'y_n^{(n-1)})$$

Substituting all $y_p^{(j)}$, $0 \le j \le n$ into Eq.(1) to get

$$(u_1 y_1^{(n)} + \dots + u_n y_n^{(n)}) + (u_1'y_1^{(n-1)}) + \dots + u_n'y_n^{(n-1)}) +$$

$$p_{n-1}\,(u_1 y_1^{(n-1)} + \dots + u_n y_n^{(n-1)}) +$$

$$\vdots$$

$$p_0\,(u_1 y_1 + \dots + u_n\, y_n) = r\,(x)$$

$$\Rightarrow\ u_1\,(y_1^{(n)} + p_{n-1}\,y_1^{(n-1)} + \ldots + p_0\,y_1) +$$

$$u_2\,(y_2^{(n)} + p_{n-1}\,y_2^{(n-1)} + \ldots + p_0\,y_2) +$$

$$\vdots$$

$$u_n\,(y_n^{(n)} + p_{n-1}\,y_n^{(n-1)} + \ldots + p_0\,y_n^{(n-1)}) +$$

$$u_1'y_1^{(n-1)} + \ldots + u_n'\,y_n^{(n-1)} = r\,(x)$$

$$\Rightarrow\ u_1'y_1^{(n-1)} + \ldots + u_n'y_n^{(n-1)} = r\,(x) \quad \text{(the last condition)}$$

$$\Rightarrow \begin{cases} u'_1 y_1 + \quad \ldots \quad + u'_n y_n = 0 \\ u'_1 y'_1 + \quad \ldots \quad + u'_n y'_n = 0 \\ \qquad\qquad \ldots \\ u'_1 y_1^{(n-2)} + \ldots + u'_n y_n^{(n-2)} = 0 \\ u'_1 y_1^{(n-1)} + \ldots + u'_n y_n^{(n-1)} = r(x) \end{cases}$$

$$\Rightarrow [W]\begin{bmatrix} u_1' \\ u_2' \\ \ldots \\ \ldots \\ u_n' \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ \ldots \\ 0 \\ r \end{bmatrix} \qquad \left(\begin{vmatrix} ta_{11} & a_{12} & \ldots & a_{1n} \\ \ldots & & & \ldots \\ ta_{n1} & a_{n2} & \ldots & a_{nn} \end{vmatrix} = t \begin{vmatrix} a_{11} & a_{12} & \ldots & a_{1n} \\ \ldots & & & \ldots \\ a_{n1} & a_{n2} & \ldots & a_{nn} \end{vmatrix} \right)$$

$$\Rightarrow\ u_j' = \frac{W_j\, r}{W} \qquad \text{(Cramer's rule)}$$

$$\Rightarrow\ u_j = \int \frac{W_j}{W}\, r\, dx$$

$$\Rightarrow\ y_p(x) = \sum y_i(x) \int \frac{W_i}{W}\, r(x)\, dx.$$

❁ Problems of Section 3.3.