2. Second-order Linear Ordinary Differential Equations

2.1 Homogeneous linear *ODEs*

- A linear second-order *DE* is formed of

$$y'' + p(x)y' + q(x)y = r(x)$$

- If $r(x) \equiv 0$ (*i.e.*, $r(x) = 0$ for all x considered), then the *DE* is called homogeneous.

  If $r(x) \not\equiv 0$, the *DE* is called nonhomogeneous.

- The functions p, q and r are called the coefficients of the equation.

- A solution of a second-order *DE* on some open interval $a < x < b$ is a function $y(x)$ that satisfies the *DE*, for all x in that interval.

- At first we discuss the properties of solutions of second-order linear *DE*, then we consider the solving strategies.
- Theorem 1 (Fundamental Theorem for homogeneous linear 2nd-order *DE*)
  For a homogeneous linear 2nd-order *DE*, any linear combination of two solutions on an open interval I is again a solution on I. In particular, for such an equation, sums and constant multiples of solutions are again solutions.

Proof.

Let y_1 and y_2 be solutions of $y'' + py' + qy = 0$ on I.

Substituting $y = c_1y_1 + c_2y_2$ into the *DE*, we get

$$y'' + py' + qy = (c_1y_1 + c_2y_2)'' + p(c_1y_1 + c_2y_2)' + q(c_1y_1 + c_2y_2)$$
$$= c_1y_1'' + c_2y_2'' + c_1py_1' + c_2py_2' + c_1qy_1 + c_2qy_2$$
$$= c_1 (y_1'' + py_1' + qy_1) + c_2 (y_2'' + py_2' + qy_2) = 0.$$

- Ex.
  $y_1 = e^{x}$ and $y_2 = e^{-x}$ are two solutions of $y'' - y = 0$
  $y = -3y_1 + 8y_2 = -3e^{x} + 8e^{-x}$ is a solution of $y'' - y = 0$
  since $(-3e^{x} + 8e^{-x}) - (-3e^{x} + 8e^{-x}) = 0$.
- **Caution!** The linear combination solutions does not hold for nonhomogeneous or nonlinear *DE*.
- Ex.2. $y_1 = 1 + \cos x$ and $y_2 = 1 + \sin x$ are solutions of nonhomogeneous *DE* $y'' + y = 1$.
  but $2(1 + \cos x)$ and $(1 + \cos x) + (1 + \sin x)$ are not solutions of $y'' + y = 1$.
- Ex.3. $y_1 = x^2$ and $y_2 = 1$ are solutions of nonlinear *DE* $y''y - xy' = 0$, but $-x^2$ and $x^2 + 1$ are not solution of $y''y - xy' = 0$.

- Definition

  A general solution of a 1st-order *DE* involved one arbitrary constant, a general solution of a 2nd-order *DE* will involve two arbitrary constant, and is of the form

  $$y = c_1 y_1 + c_2 y_2 \ ,$$

  where y_1 and y_2 are linear independent and called a basis of the *DE*.

- Definition

  A particular solution is obtained by specifying c_1 and c_2.

- Two function y_1(x) and y_2(x) are linear independent, if

  $$c_1 y_1(x) + c_2 y_2(x) = 0 \quad \text{implies} \quad c_1 = c_2 = 0 \ .$$

  In other words, two functions are linear independent if and only if they are not proportional.

- Wronskian test for linear independent

  Let $y_1(x)$ and $y_2(x)$ be solutions of $y'' + py' + qy = 0$ on an interval *I.*

  Let $W[y_1, y_2] = \begin{vmatrix} y_1 & y_2 \\ y_1' & y_2' \end{vmatrix} = y_1 y_2' - y_1' y_2$ for x in I.

  Then 1. Either $W[y_1, y_2] = 0$ for all x in I

  or $W[y_1, y_2] \neq 0$ for all x in I .

  2. $y_1(x)$ and $y_2(x)$ are linear independent on I

  $\Leftrightarrow W[y_1, y_2] \neq 0$ for some x in I.

  The proof will be given later.

- Find a basis if one solution is known (reduction of order)

Let y_1 be a solution of $y'' + py' + qy = 0$ on some interval I.
Substitute $y_2 = uy_1$ into the *DE* to get $y_2' = u'y_1 + uy_1'$ and
$y_2'' = u''y_1 + 2u'y_1' + uy_1''$.

Then $u''y_1 + 2u'y_1' + uy_1'' + p(u'y_1 + uy_1') + quy_1 = 0$

$$\Rightarrow u''y_1 + u'(2y_1' + py_1) + u(y_1'' + py_1' + qy_1) = 0$$

$$\Rightarrow u''y_1 + u'(2y_1' + py_1) = 0$$

Let $v = u'$

$$\Rightarrow v' + v(\frac{2y_1'}{y_1} + p) = 0$$

$$\Rightarrow \frac{v'}{v} = -\frac{2y_1'}{y_1} - p$$

$$\Rightarrow \ln|v| = -2\ln|y_1| - \int p dx$$

By taking exponents, we have

$$v = \frac{1}{y_1^2} e^{-\int p dx} \neq 0$$

$$\Rightarrow y_2 = uy_1 = y_1 \int v dx$$

Since $\int v\, dx = u$ can't be a constant, y_1 and y_2 form a basis of solution.

- Ex. Find a basis of solution for the *DE*

$$x^2y'' - xy' + y = 0$$

Solution.

One solution is $y_1 = x$

$$v = \frac{1}{x^2} e^{-\int (-\frac{1}{x})dx} = \frac{1}{x^2} e^{\ln x} = \frac{1}{x}$$

$$y_2 = x \int \frac{1}{x} dx = x \ln x, \quad x > 0.$$

- An initial value problem now consists of $y'' + py' + qy = 0$ and two initial conditions $y(x_0) = k_0$, $y'(x_0) = k_1$.
  The conditions are used to determine the two arbitrary constants c_1 and c_2 in the general solution.

- Ex.

  Solve the initial value problem

  $$y'' - y = 0,\ y(0) = 4 ,\ y'(0) = -2$$

  $y = c_1 e^{x} + c_2 e^{-x}$

  $y' = c_1 e^{x} - c_2 e^{-x}$

  $$\Rightarrow \begin{cases} y(0) = c_1 + c_2 = 4 \\ y'(0) = c_1 - c_2 = -2 \end{cases} \quad \Rightarrow \begin{cases} c_1 = 1 \\ c_2 = 3 \end{cases}$$

  $\Rightarrow$ solution $y = e^{x} + 3e^{-x}$.

- Problems of Section 2.1.

2.2 Homogeneous linear *ODEs* with constant coefficients

$$y'' + ay' + by = 0,$$

where coefficients a and b are constant.

Solution.

Assume $y = e^{\lambda x}$ and substituting it into the original *DE* to get

$$(\lambda^2 + a\lambda + b)\, e^{\lambda x} = 0$$

The equation $\lambda^2 + a\lambda + b = 0$ is called the characteristic equation of the *DE*. Its roots are

$$\lambda_1 = \frac{1}{2}\left(-a + \sqrt{a^2 - 4b}\right) \text{ and } \lambda_2 = \frac{1}{2}\left(-a - \sqrt{a^2 - 4b}\right) .$$

Then the solutions are $e^{\lambda_1 x}$ and $e^{\lambda_2 x}$.

Consider three cases:

case 1. two distinct real roots if $a^2 - 4b > 0$.

case 2. a real double root if $a^2 - 4b = 0$.

case 3. complex conjugate roots if $a^2 - 4b < 0$.

- *Case* 1 (Two distinct real roots λ_1 and λ_2)

  $y_1 = e^{\lambda_1 x}$ and $y_2 = e^{\lambda_2 x}$ are linear independent.

  The solution is just $y = c_1 e^{\lambda_1 x} + c_2 e^{\lambda_2 x}$.

- Ex.2. Solving $y'' + y' - 2y = 0$ with $y(0) = 4$ and $y'(0) = -5$.

  Solution.

  The characteristic equation $\lambda^2 + \lambda - 2 = 0$, it's roots are

  $\lambda_1 = \frac{1}{2}\left(-1+\sqrt{9}\right) = 1$ and $\lambda_2 = \frac{1}{2}\left(-1-\sqrt{9}\right) = -2$

  Then $y = c_1 e^{x} + c_2 e^{-2x}$

  Since $\left.\begin{aligned} y(0) &= 4 = c_1 + c_2 \\ y'(0) &= -5 = c_1 - 2c_2 \end{aligned}\right\}$

  $\Rightarrow$ $c_1 = 1$ and $c_2 = 3$

  $\Rightarrow$ $y = e^{x} + 3e^{-2x}$.

- *Case* 2 (Real double root $\lambda = -\frac{a}{2}$)

$x^3 - 6x^2 + 11x - 6 = 0$ hasroot1, whatarethe otherroots?

  Assume $y_1 = e^{-\frac{a}{2}x}$

  Use the "method of reduction of order" to find another solution.

step 1. Set $y_2 = uy_1$, and try to determine the function u such that y_2 is an independent solution.

$$y_2' = u'y_1 + uy_1' \text{ and } y_2'' = u''y_1 + 2u'y_1' + uy_1''$$

step 2. Substituting y_2 in the original *DE*

$$(u''y_1 + 2u'y_1' + uy_1'') + a\,(u'y_1 + uy_1') + buy_1 = 0$$

$\Rightarrow$ $u''y_1 + u'(2y_1' + ay_1) + u\,(y_1'' + ay_1' + by_1) = 0$

Since y_1 is a solution, $y_1'' + ay_1' + by_1 = 0$

Since $2y_1' = 2(-\frac{a}{2})e^{-\frac{a}{2}x} = -a\,e^{-\frac{a}{2}x} = -a\,y_1$

$\Rightarrow$ $2y_1' + ay_1 = 0$ $\Rightarrow$ $u''y_1 = 0$

$\Rightarrow$ $u'' = 0$

$\Rightarrow$ $u' = c$ (This is the order reduced equation.)

$\Rightarrow$ $u = c_1 x + c_2$

Simply taking $c_1 = 1$ and $c_2 = 0$

We get $y_2 = xy_1$, y_1 and y_2 are linear independent.

Thus the general solution

$$y = (c_1 + c_2 x)\, e^{-\frac{a}{2}x}.$$

- Warning.

  If λ is a simple root of $y'' + py' + qy = 0$,

  then $(c_1 + c_2 x)\, e^{\lambda x}$ is not a solution of the *DE*.

  (Explanation on page 19)

- Ex. (omitted)

- *Case* 3 (Complex roots)

  Complex exponential function

$$e^{ix} = 1 + ix + \frac{(ix)^2}{2!} + \frac{(ix)^3}{3!} + \frac{(ix)^4}{4!} + \frac{(ix)^5}{5!} + \ldots$$

$$= (1 - \frac{x^2}{2!} + \frac{x^4}{4!} + \ldots) + i(x - \frac{x^3}{3!} + \frac{x^5}{5!} + \ldots)$$

$$= \cos x + i \sin x$$

Euler formula
$e^{\pm ix} = \cos x \pm i \sin x$

$$\lambda = -\tfrac{1}{2}a \pm \tfrac{1}{2}\sqrt{a^2 - 4b}\,, \text{ where } a^2 - 4b < 0$$

$$\Rightarrow \lambda = -\tfrac{1}{2}a \pm i\omega \quad \text{where} \quad \omega = \sqrt{b - \tfrac{1}{4}a^2} > 0$$

Then $e^{\lambda_1 x}$ and $e^{\lambda_2 x}$ are complex solutions of the original *DE*.

The solution can be derived further.

$z_1 = e^{\lambda_1 x} = e^{-\frac{a}{2}x} e^{i\omega x} = e^{-\frac{a}{2}x}$ ($\cos \omega x + i \sin \omega x$) and

$z_2 = e^{\lambda_2 x} = e^{-\frac{a}{2}x} e^{-i\omega x} = e^{-\frac{a}{2}x}$ ($\cos \omega x - i \sin \omega x$).

A basis of real solution of the *DE* is

$$y_1 = \frac{z_1 + z_2}{2} = e^{-\frac{a}{2}x}\cos\omega x \quad \text{and} \quad y_2 = \frac{z_1 - z_2}{2i} = e^{-\frac{a}{2}x}\sin\omega x.$$

Testing for linear independence of y_1 and y_2

$$\begin{vmatrix} e^{-\frac{1}{2}ax}\cos\omega x & e^{-\frac{1}{2}ax}\sin\omega x \\ -\frac{1}{2}a\,e^{-\frac{1}{2}ax}\cos\omega x - \omega\,e^{-\frac{1}{2}ax}\sin\omega x & -\frac{1}{2}a\,e^{-\frac{1}{2}ax}\sin\omega x + \omega\,e^{-\frac{1}{2}ax}\cos\omega x \end{vmatrix}$$

$$= 2e^{-ax}(\cos\omega x)^2 + \omega\,e^{-ax}(\sin\omega x)^2 = \omega\,e^{-ax} \neq 0.$$

Then the general solution becomes

$$y = e^{-\frac{a}{2}x}(c_1\cos\omega x + c_2\sin\omega x).$$

- Summary of *cases* 1, 2, and 3

case 1. real $\lambda_1 \neq \lambda_2 \Rightarrow y = c_1e^{\lambda_1 x} + c_2e^{\lambda_2 x}$.

case 2. double real $\lambda_1 = \lambda_2 = -\frac{1}{2}a \Rightarrow y = (c_1 + c_2x)e^{-\frac{ax}{2}}$.

case 3. complex conjugate

$$\lambda_1 = -\frac{1}{2}a + i\omega,$$

$$\lambda_2 = -\frac{1}{2}a - i\omega, \quad \omega = \sqrt{b - \frac{1}{4}a^2}$$

$$\Rightarrow y = e^{-\frac{1}{2}ax}(c_1\cos\omega x + c_2\sin\omega x).$$

- Initial value problem = a *DE* + initial conditions.
  For example, $y'' + 2y' + 5y = 0$, $y(0) = 1$, $y'(0) = 5$.

- Ex. (Boundary value problem)
  A boundary value problem = a *DE* + boundary conditions
  $y'' + y = 0$ and two boundary conditions
  $y(0) = 3$
  $y(\pi) = -3$.
  Characteristic function $\lambda^2+1 = 0$, $\lambda = \pm i$
  $y = c_1 \cos x + c_2 \sin x.$
  Since $y(0) = 3 = c_1 \Rightarrow c_1 = 3$
  $y(\pi) = -3 = -c_1 \Rightarrow c_1 = 3$
  Thus $y = 3\cos x + c_2 \sin x.$

- Problems of Section 2.2.

2.3 Differential operator

- Introduce a method for solving *DE,* say operational method.
  Let *D* denote differentiation with respect to *x*, write $Dy = y'$.
  D is an operator, it transforms *y* into its derivative *y'*.
  For example,
  $D(x^2) = 2x$, $D(\sin x) = \cos x$
  $D(Dy) = Dy' = y''$,
  $D^3 y = y'''$, …….

- Define another operator *L*, $L = P(D) = D^2 + aD + b$, call a 2nd-order differential operator. Here *a* and *b* are constant, *P* means polynomial, and *L* means linear.
  $L[y] = P(D)[y] = (D^2 + aD + b)\, y = y'' + ay' + by.$

- L is linear operator; *i.e.*, $L[\alpha y + \beta z] = \alpha L[y] + \beta L[z]$,

  $P(D)[e^{\lambda_1 x}] = (\lambda_1^2 + a\lambda_1 + b)\, e^{\lambda_1 x} = P(\lambda_1)\, e^{\lambda_1 x} = 0$

  $e^{\lambda_1 x}$ is a solution of $y'' + ay' + by = 0$

  $\Updownarrow$

  $P(\lambda_1) = 0$ (λ_1 is a root of $\lambda^2 + a\lambda + b = 0$)

  If $P(\lambda)$ has two different roots, we obtain a basis.
  If $P(\lambda)$ has a double root, we need a second independent solution.
  Differentiate both sides of $P(D)[e^{\lambda x}] = P(\lambda)\, e^{\lambda x}$ *w.r.t.* λ to get

  $P(D)[x e^{\lambda x}] = P'(\lambda)\, e^{\lambda x} + P(\lambda)\, x e^{\lambda x}$

  $\Rightarrow P(D)[x e^{\lambda_1 x}] = P'(\lambda_1)\, e^{\lambda_1 x} + P(\lambda_1)\, x e^{\lambda_1 x} = 0$

  $x e^{\lambda_1 x}$ is a solution of $y'' + ay' + by = 0$

  $\Updownarrow$

  $P'(\lambda_1) = 0$ and $P(\lambda_1) = 0$

  (That is, λ_1 is a double root of $\lambda^2 + a\lambda + b = 0$)

  $P(\lambda) / (\lambda - \lambda_1) = P'(\lambda)$ (reduction of order)

- How to use differential operator to solve *DE* ?

- Ex. Solving $y'' + y' - 6y = 0$

$$\Rightarrow (D^2 + D - 6)\, y = 0$$

$$\Rightarrow (D + 3)(D - 2)\, y = 0$$

$$\Rightarrow \begin{cases} (D+3)\, y = 0 \\ (D-2)\, y = 0 \end{cases}$$

$$\Rightarrow \begin{cases} y' + 3y = 0 \\ y' - 2y = 0 \end{cases}$$

$$\Rightarrow \begin{cases} \lambda + 3 = 0 \\ \lambda - 2 = 0 \end{cases}$$

$$\Rightarrow \begin{cases} \lambda_1 = -3 \\ \lambda_2 = 2 \end{cases}$$

$$\Rightarrow \begin{cases} y_1 = e^{-3x} \\ y_2 = e^{2x} \end{cases}$$

$$\Rightarrow y = c_1 e^{-3x} + c_2 e^{2x}.$$

- Problems of Section 2.3.

2.5 Euler-Cauchy equation

$x^2y'' + axy' + by = 0$ ………………………………… (1)

Solution.

Substituting $y = x^m$ into Eq.(1), to get

$$x^2 m(m-1)x^{m-2} + axmx^{m-1} + bx^m = 0$$

$$\Rightarrow m(m-1) + am + b = 0 \quad \text{if } x \neq 0$$

$$\Rightarrow m^2 + (a-1)m + b = 0 \quad \text{(auxiliaryequation)}$$

Consider three cases

case 1. Distinct real roots $m = \frac{1}{2}\left[(1-a) \pm \sqrt{(a-1)^2 - 4b}\right]$

Then the general solution is $y = c_1x^{m_1} + c_2x^{m_2}$.

case 2. Double root

$$m = \frac{1}{2}(1-a) \Rightarrow y_1 = x^{\frac{1-a}{2}}$$

using the method of reduction of order to find another solution. Substituting $y_2 = uy_1$ into Eq.(1). We obtain

$$x^2(u''y_1 + 2u'y_1' + uy_1'') + ax(u'y_1 + uy_1') + buy_1 = 0$$

$$\Rightarrow \underline{u''x^2y_1 + u'x(2xy_1' + ay_1)} + u(x^2y_1'' + axy_1' + by_1) = 0$$

Since y_1 is a solution, $x^2y_1'' + axy_1' + by_1 = 0$

Since $y_1 = x^{\frac{1-a}{2}}$, $2xy_1' + ay_1 = 2x\left(\frac{1-a}{2}\right)x^{\frac{1-a-2}{2}} + a\,x^{\frac{1-a}{2}}$

$$= (1-a)x^{\frac{1-a}{2}} + a\,x^{\frac{1-a}{2}}$$

$$= x^{\frac{1-a}{2}} = y_1$$

$\Rightarrow u''x^2y_1 + u'xy_1 = 0$

$\Rightarrow u''x + u' = 0.$

$$\Rightarrow \frac{u''}{u'} = -\frac{1}{x}$$
$$\Rightarrow \ln|u'| = -\ln x, \quad \text{for } x > 0$$
$$\Rightarrow u' = \frac{1}{x}$$
$$\Rightarrow u = \ln x$$
$$\Rightarrow y_2 = \ln x\, x^m, \quad m = \tfrac{1-a}{2}.$$

Thus the general solution $y = (c_1 + c_2 \ln x)\, x^{\frac{1-a}{2}}$.

The following derivation is wrong, since $\ln x, \quad x < 0$ is not defined.

$$\ln|u'| = \ln|x|, \quad \text{if } x < 0$$
$$\Rightarrow u' = x$$
$$\Rightarrow u = \frac{x^2}{2}$$
$$\Rightarrow y_2 = x^{m+2} \quad \text{It is wrong.}$$

case 3. Complex conjugate roots

$$m = \frac{1-a}{2} \pm \frac{\sqrt{(a-1)^2 - 4b}}{2}, \quad \text{where } (a-1)^2 - 4b < 0$$
$$\Rightarrow m = \frac{1-a}{2} \pm i\frac{\sqrt{4b - (a-1)^2}}{2}$$
$$\Rightarrow \begin{cases} m_1 = \mu + i\nu \\ m_2 = \mu - i\nu \end{cases}$$
$$\Rightarrow \begin{cases} z_1 = x^{m_1} = x^{\mu} x^{i\nu} = x^{\mu} e^{i\nu \ln x} \\ \quad = x^{\mu}\left[\cos(\nu \ln x) + i \sin(\nu \ln x)\right] \\ z_2 = x^{m_2} = x^{\mu} x^{-i\nu} = x^{\mu} e^{-i\nu \ln x} \\ \quad = x^{\mu}\left[\cos(\nu \ln x) - i \sin(\nu \ln x)\right] \end{cases}$$
$$\Rightarrow \begin{cases} y_1 = x^{\mu} \cos(\nu \ln x) \\ y_2 = x^{\mu} \sin(\nu \ln x) \end{cases}$$

The general solution $y = (a \cos(\nu \ln x) + b \sin(\nu \ln x))\, x^{\mu}$.

- Problems of Section 2.5.

2.6 Existence and uniqueness theory

- Theorem 1 (Existence and uniqueness theory for initial value problems)

  For an initial values problem

  $y'' + py' + qy = 0$

  $y(x_0) = k_0,\ y'(x_0) = k_1,$

  if $p(x)$ and $q(x)$ are continues functions on some open interval I and x_0 is in I, then the initial value problem has a unique solution $y(x)$ on the interval I.

- We will discuss the existence of a general solution for a *DE*.
- For a 2nd-order *DE*, existing a general solution $c_1y_1 + c_2y_2$ means y_1 and y_2 are linear independent; that is,

  $k_1y_1 + k_2y_2 = 0 \Rightarrow k_1 = k_2 = 0.$

- We know that the Wronskian $\begin{vmatrix} y_1 & y_2 \\ y_1' & y_2' \end{vmatrix} \neq 0 \Leftrightarrow y_1$ and y_2 are linear independent.

- Now we want to prove the statement
- Theorem 2 (Wronskian test for linear independent)

  Let $y_1(x)$ and $y_2(x)$ be solution of $y'' + py' + qy = 0$ on an interval I

  $$W[y_1,y_2] = \begin{vmatrix} y_1 & y_2 \\ y_1' & y_2' \end{vmatrix} = y_1y_2' - y_1'y_2 \quad \text{for all } x \text{ in } I$$

  Then (*a*) Either $W[y_1,y_2] = 0$ for all x in I,

  or $W[y_1,y_2] \neq 0$ for all x in I,

  (*b*) y_1 and y_2 are linear independent on I

  $\Leftrightarrow W[y_1,y_2] \neq 0$ for some x in I .

Proof

(*a*) Calculate $W' = y_1'y_2' + y_1y_2'' - y_1''y_2 - y_1'y_2'$

$= y_1y_2'' - y_1''y_2$

Since $\begin{cases} y_1'' + py_1' + qy_1 = 0 \\ y_2'' + py_2' + qy_2 = 0 \end{cases}$

$$\Rightarrow \begin{cases} -y_2y_1'' - py_2y_1' - qy_2y_1 = 0 \\ y_1y_2'' + py_1y_2' + qy_1y_2 = 0 \end{cases}$$

$\Rightarrow$ $-y_2y_1'' + y_1y_2'' + p\,(-y_2y_1' + y_1y_2') = 0$

$\Rightarrow$ $W' + p\,W = 0$ (A linear 1st-order *DE*)

$\Rightarrow$ $W = ce^{-\int p dx}$

$$\Rightarrow \begin{cases} \text{if } c = 0 \Rightarrow W = 0 \quad \text{for all } x \text{ in } I, \\ \text{if } c \neq 0 \Rightarrow W \neq 0 \quad \text{for all } x \text{ in } I \end{cases}$$

(*b*) " $\Leftarrow$ " (independent $\Leftarrow$ $W \neq 0$)

Prove by showing "dependent $\Rightarrow W = 0$".

If y_1 and y_2 are linear dependent, then $y_2 = ky_1$ for some constant *k.* Thus

$W[y_1,y_2] = y_1y_2' - y_1'y_2 = ky_1y_1' - ky_1y_1' = 0$.

" $\Rightarrow$ " (independent $\Rightarrow W \neq 0$)

prove by showing "$W = 0$ $\Rightarrow$ dependent".

Assume $y_1 \neq 0$, $y_2 \neq 0$. Consider the linear system

$$\begin{cases} k_1y_1(x_0) + k_2y_2(x_0) = 0 \\ k_1y_1'(x_0) + k_2y_2'(x_0) = 0, \end{cases}$$

where k_1 and k_2 are unknown.

$$\Rightarrow \begin{bmatrix} y_1(x_0) & y_2(x_0) \\ y_1'(x_0) & y_2'(x_0) \end{bmatrix} \begin{bmatrix} k_1 \\ k_2 \end{bmatrix} = \mathbf{0}$$

By Cramer's theorem, if $\begin{vmatrix} y_1 & y_2 \\ y_1' & y_2' \end{vmatrix} = 0 \quad (i.e., W = 0)$,

then the system has a non-trivial solution; that is, k_1 and k_2 are not both zero. Using these k_1 and k_2 to construct a function $y(x) = k_1y_1(x) + k_2y_2(x)$, then $y(x) = k_1y_1 + k_2y_2$ is a solution of $y'' + py' + qy = 0$, and y satisfies the initial conditions $y(x_0) = 0$ and $y'(x_0) = 0$

By the existence and uniqueness theorem, we known that the solution is unique such that $k_1y_1 + k_2y_2 = 0$ on *I*. Since k_1 and k_2 are not both zero, y_1 and y_2 are linear dependent.

- A general solution of $y'' + py' + qy = 0$ includes all solutions; that is, exist a solution of form $c_1y_1 + c_2y_2$.
- Theorem 3 (Existence of a gereral solution)

  If the coefficients $p(x)$ and $q(x)$ of $y'' + py' + qy = 0$ are continuous on some open interval *I*, then the *DE* has a general solution on *I*.
- Theorem 4 (General solution)

  If the coefficients $p(x)$ and $q(x)$ of $y'' + py' + qy = 0$ are continuous on some open interval, then every solution $y = y(x)$ on *I* is of the form $y(x) = c_1y_1(x) + c_2y_2(x)$.
- Problems of Section 2.6.

2.7 Nonhomogeneous *ODEs*

- Given a nonhomogeneous equation

$$y'' + p(x)y' + q(x)y = r(x)\ ,\ r(x)\neq 0 \quad \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots \quad (1)$$

Consider the corresponding homogenous equation

$$y'' + p(x)y' + q(x)y = 0 \quad \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots \quad (2)$$

- Definition

A general solution of the nonhomogeneous equation (1) on some open interval *I* is a solution of the form

$$y(x) = y_h(x) + y_p(x) \quad \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots \quad (3)$$

where $y_h(x) = c_1y_1(x) + c_2y_2(x)$ is a general solution of the homogeneous equation (2) on *I* and $y_p(x)$ is any solution of Eq.(1) on *I* containing no arbitrary constants and called a particular solution of equation (1).

❁ Theorem 1 (Relation between solutions of Eqs.(1) and (2))

(*a*) The difference of two solutions of Eq.(1) on some open interval *I* is a solution of Eq.(2) on *I.*

(*b*) The sum of a solution of Eq.(1) on *I* and a solution of Eq.(2) on *I* is a solution of Eq.(1) on *I*.

Proof.

(*a*) Let $y_1(x)$ and $y_2(x)$ be two solutions of $y'' + py' + qy = r$.

$\Rightarrow\ y_1'' + py_1' + qy_1 = r$ and $y_2'' + py_2' + qy_2 = r$

$\Rightarrow\ (y_1'' + py_1' + qy_1) - (y_2'' + py_2' + qy_2) = r - r = 0$

$\Rightarrow\ (y_1 - y_2)'' + p(y_1 - y_2)' + q(y_1 - y_2) = 0.$

(*b*) Let y_1 be a solution of Eq.(1), then $y_1'' + py_1' + qy_1 = r$, and

y_2 be a solution of Eq.(2), then $y_2'' + py_2' + qy_2 = 0$

$\Rightarrow (y_1 + y_2)'' + p(y_1 + y_2)' + q(y_1 + y_2) = r.$

❁ Theorem2 (General solution)

Suppose that $p(x)$, $q(x)$ and $r(x)$ in Eq.(1) are continuous on some open interval *I*. Then every solution of Eq.(1) on *I* is obtained by assigning suitable values to the arbitrary constant in a general solution (3) of Eq.(1) on *I*.

❁ Conclusion

a general solution of nonhomogeneous equation

||

a general solution of the corresponding homogeneous equation

\+

a particular solution of the nonhomogeneous equation.

Solution by undetermined coefficients

- A simple method for finding the particular solution of

  $y'' + py' + qy = r$ ……………………………………………… (1)

- Principle:
  Choose for y_p a form similar to that of $r(x)$ and involving unknown coefficient to be determined by substituting that choice for y_p into Eq.(1).

- How to choose the solution form:

  (*a*) Basic rule: Table 2.1 on page 80

$r(x)$	$y_p(x)$
$k\,e^{vx}$	$c\,e^{vx}$
$k\,x^n$	$k_n x^n + k_{n-1}x^{n-1} + \ldots + k_1 x + k_0$
$k\cos\omega x$, $k\sin\omega x$	$m\cos\omega x + n\sin\omega x$
$k\,e^{\alpha x}\cos\omega x$, $k\,e^{\alpha x}\sin\omega x$	$e^{\alpha x}(m\cos\omega x + n\sin\omega x)$

(*b*) Modification rule:
If $cr(x)$ = a solution of the corresponding homogeneous equation, then choose $y_p(x) = kxr(x)$; or $y_p(x) = kx^2r(x)$, if $r(x)$ corresponds to a double root of the characteristic equation, …
Note that $xr(x)$ is linearly independent to $xr(x) + x^2r(x)$.

(*c*) Sum rule:
If $r(x)$ is a linear combination of functions listed in the left column of Table 2.1, then $y_p(x)$ is taken as linear combination of the corresponding functions listed in the right column of Table 2.1.

- The properties of the method

  (*a*) The method corrects itself in the same that a false choice of y_p.
  (*b*) One choice with too few terms will lead to a contradiction.
  (c) A choice of too many terms will give a correct result, with superfluous coefficients coming out zero.

- Ex.1 ~ 4.

  (1) $r(x) = 8x^2 \Rightarrow y_p(x) = k_2x^2 + k_1x + k_0$.

  If you choose $y_p(x) = k_2x^2$ would fail, why ?

  For example, $y'' + 4y = 8x^2$

  $2k_2 + 4k_2x^2 = 8x^2$ (contradiction)

  If you choose $y_p(x) = k_2x^2 + k_1x + k_0$

  $2k_2 + 4k_2x^2 + 4k_1x + 4k_0 = 8x^2 \Rightarrow k_2 = 2, k_1 = 0, k_0 = -1$

  (2) $r(x) = e^x$, $y_h(x) = c\, e^x \Rightarrow y_p(x) = cxe^x$.

  (3) $r(x) = e^x + x$, $y_h(x) = (c_1 + c_2x)\, e^x \Rightarrow$

  $y_p(x) = cx^2 e^x + k_1x + k_o$.

  (4) $r(x) = 16e^x + \sin 2x$, $y_h(x) = e^{-x}(A \cos 2x + B \sin 2x)$

  $\Rightarrow y_p(x) = ke^x + c_1 \cos 2x + c_2 \sin 2x.$

- Problems of Section 2.7.

2.10 Solution by variation of parameters

- A general method to solve nonhomogeneous *DE* without special right side $r(x)$. For example

$$y'' - 2y' + y = x^{\frac{3}{2}} e^x$$

- For the *DE* $y'' + p(x)\, y' + q(x)\, y = r(x)$. The method "variation of parameters" gives a particular solution y_p in the form

$$y_p(x) = -y_1 \int \frac{y_2 r}{w}\, dx + y_2 \int \frac{y_1 r}{w}\, dx,$$

where y_1 and y_2 form a basis of solutions of the corresponding equation

$$y'' + p(x)\, y' + q(x)\, y = 0$$

and $w = y_1y_2' - y_1'y_2$ is the Wronskian of y_1 and y_2.

✿ Ex. Solving $y'' + y = \sec x$.

$y_1 = \cos x$, $y_2 = \sin x$, and $W[y_1, y_2] = 1$.

$$y_p = -\cos x \int \sin x \sec x \, dx + \sin x \int \cos x \sec x \, dx$$
$$= -\cos x \int \frac{\sin x}{\cos x} dx + \sin x \int \frac{\cos x}{\cos x} dx$$
$$= -\cos x \int -\left(\frac{\cos' x}{\cos x}\right) dx + x \sin x$$
$$= \cos x \ln|\cos x| + x \sin x$$
$$\Rightarrow y = y_h + y_p = c_1 \cos x + c_2 \sin x + \cos x \ln|\cos x| + x \sin x$$
$$= \left(c_1 + \ln|\cos x|\right)\cos x + \left(c_2 + x\right)\sin x.$$

✿ How to get $y_p(x) = -y_1 \int \frac{y_2 r}{W} dx + y_2 \int \frac{y_1 r}{W} dx$?

For a nonhomogeneous *DE* and the corresponding homogeneous *DE*:

$y'' + p(x)\, y' + q(x)\, y = r(x)$ ………………………………………….. (1)

and $y'' + p(x)\, y' + q(x)\, y = 0$.

If p and q are continuous in an open interval I, the homogeneous *DE* has a general solution

$y_h(x) = c_1 y_1(x) + c_2 y_2(x)$ on I.

Now we replace c_1 and c_2 by functions $u(x)$ and $v(x)$ and to determine $u(x)$ and v(x) functions such that

$y_p(x) = u(x) y_1(x) + v(x) y_2(x)$ ………………………………………. (2)

Differentiating Eq.(2) to obtain

$y_p'(x) = u' y_1 + u y_1' + v' y_2 + v y_2'$

We need two conditions to determine $u(x)$ and $v(x)$; one condition is that $y_p(x)$ satisfies Eq(1). Now we assume another condition $u' y_1 + v' y_2 = 0$.

This reduce $y_p'(x)$ to the form $y_p'(x) = uy_1' + vy_2'$ ……….. (3)
Differentiating the equation, we have

$$y_p'' = u'y_1' + uy_1'' + v'y_2' + vy_2'' \quad \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots (4)$$

Substituting Eqs.(2), (3) and (4) into Eq.(1), we obtain

$$\underline{u'y_1' + uy_1'' + v'y_2' + vy_2''} + puy_1' + pvy_2' + quy_1 + qvy_2 = r$$

$$\Rightarrow u(y_1'' + py_1' + qy_1) + v(y_2'' + py_2' + qy_2) + u'y_1' + v'y_2' = r$$

$$\Rightarrow u'y_1' + v'y_2' = r$$

$$\Rightarrow \begin{cases} u'y_1 + v'y_2 = 0 \\ u'y_1' + v'y_2' = r \end{cases}$$

$$\Rightarrow u' = \frac{-r\,y_2}{W[y_1, y_2]} \quad \text{and} \quad v' = \frac{r\,y_1}{W[y_1, y_2]}.$$

Since $W[y_1, y_2] \neq 0$

$$\Rightarrow u = -\int \frac{r\,y_2}{w}dx \text{ and } v = \int \frac{r\,y_1}{w}dx$$

$$\Rightarrow y_p = -y_1\int \frac{r\,y_2}{w}dx + y_2\int \frac{r\,y_1}{w}dx\,.$$

❁ Problems of Section 2.10.